Student Number ....

Barker College

2011
TRIAL
HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

Staff Involved:

- **GDH**
- **MRB**
- **BHC***
- **VAB***

PM THURSDAY 4TH AUGUST
TIME: 3 HOURS

50 copies

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Make sure your Barker Student Number is on ALL pages of your answer sheets.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.

Total marks – 120

- Attempt Questions 1–8.
- ALL necessary working should be shown in every question.
- Start each question on a NEW page.
- Write on one side only of each answer page.
- Marks may be deducted for careless or badly arranged work.

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Total marks – 120
Attempt Questions 1– 8

Answer each question on a **SEPARATE** sheet of paper

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) (i) Find $\int \frac{dx}{3 - 2x - x^2}$ using partial fractions. **4**

(ii) Hence, or otherwise find $\int \frac{2 + x}{3 - 2x - x^2}\, dx$ **2**

(b) (i) Find $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$ **2**

(ii) Hence, or otherwise, find $\int \frac{1 + 2x}{\sqrt{3 - 2x - x^2}}\, dx$ **3**

(c) Find $\int \sqrt{x^2 + a^2}\, dx$ using integration by parts. **4**

End of Question 1

Marks

Question 2 (15 marks) **[START A NEW PAGE]**

(a) (i) Solve $z^3 = \sqrt{2} + \sqrt{2}\,i$, giving answers in the form $R\,cis\,\theta$. **2**

(ii) Hence prove that $\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{9\pi}{12}\right) + \cos\left(\frac{-7\pi}{12}\right) = 0$ **1**

(b) Find the locus of Z for the following:

You may give your answer as an equation or a graph, whichever you prefer.

(i) $\frac{Z - i}{Z - 2}$ is purely real. **2**

(ii) $\frac{Z - i}{Z - 2}$ is purely imaginary. **2**

(c) Let $z = \cos\theta + i\sin\theta$.

(i) Using de Moivre's Theorem and the Binomial Theorem, show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$ **3**

(ii) Hence solve:

$$32x^5 - 40x^3 + 10x = 1$$ **3**

(iii) Hence prove that:

$$\cos\left(\frac{\pi}{15}\right) . \cos\left(\frac{7\pi}{15}\right) . \cos\left(\frac{11\pi}{15}\right) . \cos\left(\frac{13\pi}{15}\right) = \frac{1}{16}$$ **2**

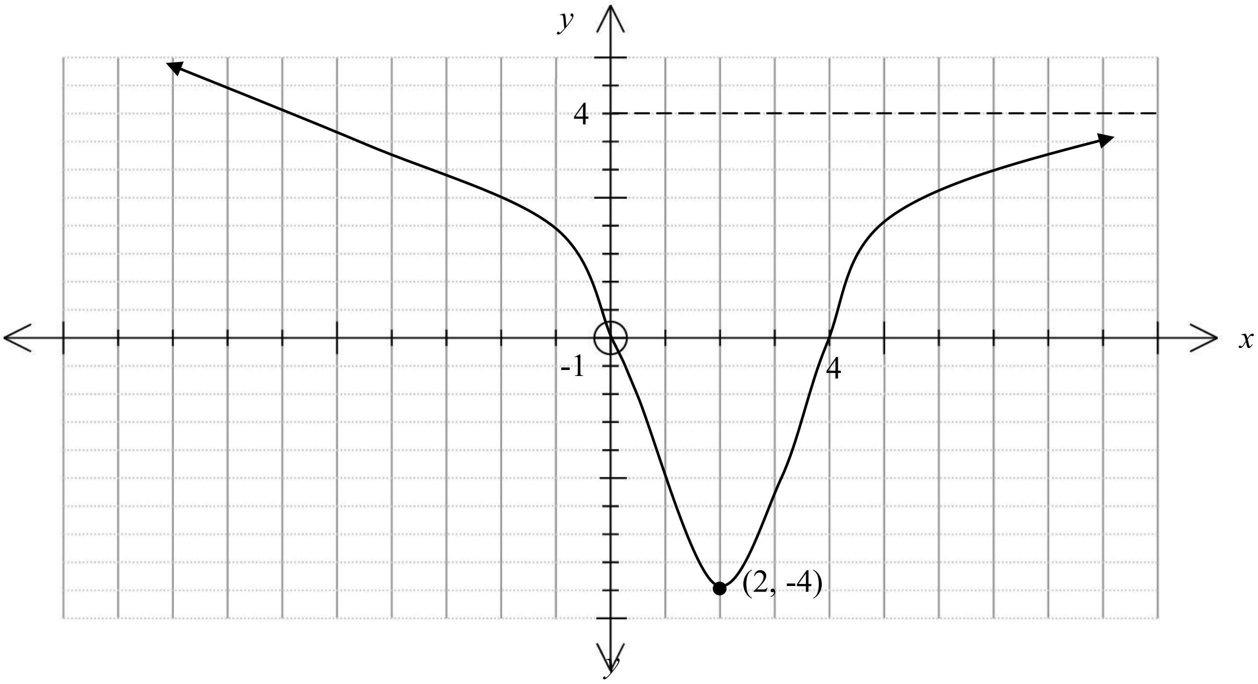
End of Question 2

Marks

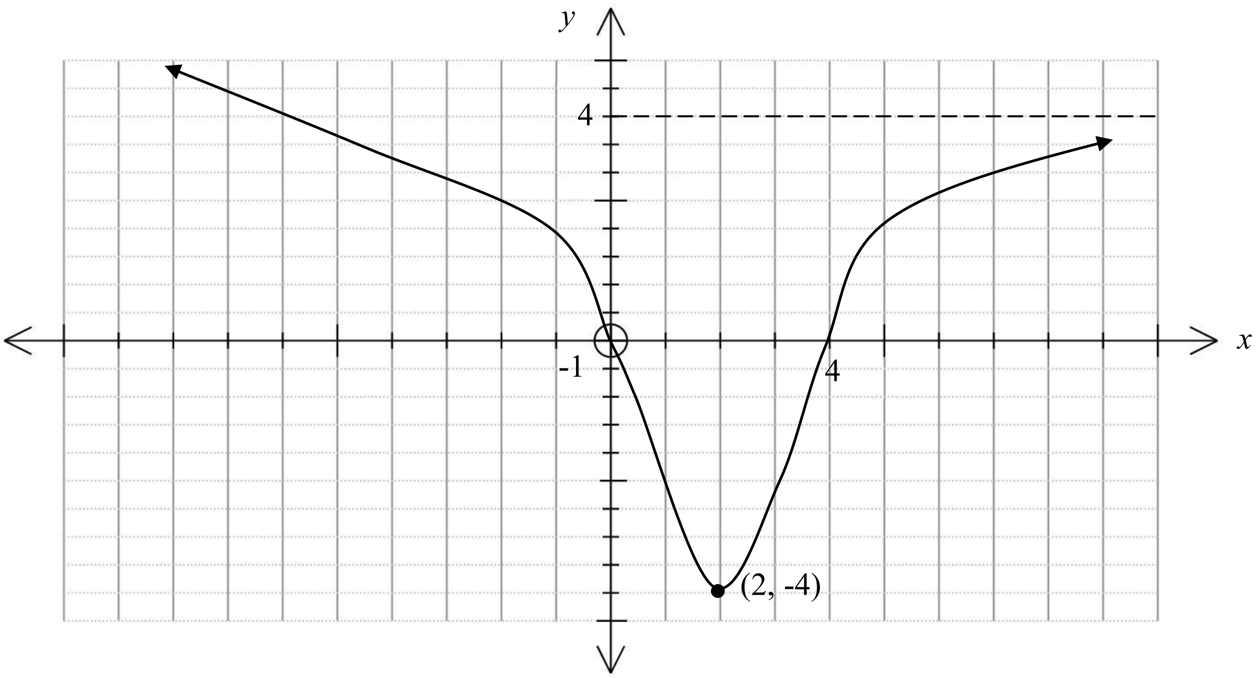
Question 3 (15 marks) **[START A NEW PAGE]**

(a) These two diagrams show the same graph of $y = f(x)$

(i)



(ii)



(i) Sketch $y = f(x^2)$ on diagram (i) above, showing x intercepts and other key features of this graph. **3**

(ii) Sketch $y = \log_e \left[f(x) \right]$ on diagram (ii) above, showing key features. **3**

DETACH THIS PAGE AND ATTACH IT TO YOUR SOLUTIONS.

Question 3 continues on page 5

Marks

Question 3 (continued)

(b) Find the ***x*-coordinates** of the points on the curve

$$2x^2 + 2xy + 3y^2 = 15$$

where the tangents to the curve are vertical. **3**

(c) (i) Sketch $y = x^2 - 2$ and $y = e^{-x}$ on the same number plane diagram. The diagram should be about one third of the page in size. **1**

(ii) Find the ***x*-coordinates** of the stationary points on $y = e^{-x}(x^2 - 2)$ **2**

(iii) Hence, sketch the graph of $y = e^{-x}(x^2 - 2)$ on the same diagram as in (i), showing the x-intercepts and other key features of the graph. **3**

End of Question 3

Marks

Question 4 (15 marks) **[START A NEW PAGE]**

(a) An ellipse has the equation $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$ and $P(x_1,\ y_1)$ is a point on this ellipse.

(i) Find its eccentricity, the coordinates of its foci, S and S^1, and the equations of its directrices. **3**

(ii) Prove that the sum of the distances SP and S^1P is independent of the position of *P*. **2**

(iii) Show that the equation of the tangent to the ellipse at *P* is

$$x_1 x + 2y_1 y = 8.$$ **3**

(iv) The tangent at $P(x_1, y_1)$ meets the directrix closest to *S* at *T*.

Prove that $\angle PST$ is a right angle. **3**

(b) The point $T\left(ct, \dfrac{c}{t}\right)$ lies on the hyperbola $xy = c^2$.

The normal at *T* meets the line $y = x$ at *R*.

Find the coordinates of *R*. **4**

End of Question 4

Marks

Question 5 (15 marks) **[START A NEW PAGE]**

(a) Given the polynomial

$$p(x) = ax^3 + bx^2 + cx + d$$

where a, b, c, d and β are integers and $p(\beta) = 0$:

(i) Prove that β divides d **2**

(ii) Hence, or otherwise, prove that the polynomial equation

$q(x) = 2x^3 - 5x^2 + 8x - 3 = 0$ does not have an integer root. **2**

(b) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 0$$
$$\alpha^2 + \beta^2 + \gamma^2 = -2$$
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-1}{10}$$

(i) Find the values of $\alpha\beta + \beta\gamma + \alpha\gamma$ and $\alpha\beta\gamma$ **3**

(ii) Hence write down a cubic equation with roots α, β and γ

in the form $ax^3 + bx^2 + cx + d = 0$ **1**

Question 5 continues on page 8

Question 5 (continued)

(c) The equation $x^3 + x^2 + 2x - 4 = 0$ has roots α, β, and γ.

(i) Evaluate $\alpha\beta\gamma$ **1**

(ii) Write an equation in the form

$$ax^3 + bx^2 + cx + d = 0$$

(A) with roots α^2, β^2 and γ^2 **3**

(B) with roots $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$ **3**

End of Question 5

Marks

Question 6 (15 marks) **[START A NEW PAGE]**

(a) Find the volume of the solid generated when the area bounded by

$y = 6 - x^2 - 3x$ and $y = 3 - x$ is revolved about the line $x = 3$. **4**

(b) (i) By rewriting

$\cos(n + 2)x$ as $\cos\{(n+1)+1\}x$,

and

$\cos nx$ as $\cos\{(n+1)-1\}x$,

show that $\cos(n + 2)x + \cos nx = 2\cos(n+1)x.\cos(x)$ **1**

(ii) Hence prove that given $u_n = \int_0^{\pi} \frac{1 - \cos nx}{1 - \cos x}\, dx$ **3**

where n is a positive integer or zero,

then,

$$u_{n+2} + u_n - 2u_{n+1} = \int_0^{\pi} \frac{2\cos(n+1)x.\{1 - \cos x\}\, dx}{1 - \cos x}$$
$$= 0$$

(iii) Evaluate u_0 and u_1 **directly**, and hence evaluate u_2 and u_3 **3**

(iv) Also show that $\int_0^{\frac{\pi}{2}} \frac{\sin^2 3\theta}{\sin^2 \theta}\, d\theta = \frac{3\pi}{2}$ **4**

End of Question 6

Marks

Question 7 (15 marks) **[START A NEW PAGE]**

(a) The acceleration due to gravity at a point outside the Earth is inversely proportional to the square of the distance from the centre of the Earth, ie. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-k}{x^2}$

(i) Neglecting air resistance, show that if a particle is projected vertically upwards with speed u from a point on the Earth's surface, its speed V in any position x is given by

$$V^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right),$$

where R is the radius of the Earth, and g is the acceleration due to gravity at the Earth's surface. **3**

(ii) Show that the greatest height *H*, **above the Earth's surface**, reached by the particle is given by

$$H = \frac{u^2 R}{2gR - u^2}$$ **2**

(iii) Prove that if the speed of projection exceeds 12 km/sec, the particle will escape the Earth's influence. (Take $R = 6400$ km and $g = 10 m/\sec^2$) **3**

Question 7 continues on page 11

Marks

Question 7 (continued)

(b) Suppose that x is a positive number less than 1, and n is a non-negative integer.

(i) Explain why

$$1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots = \frac{1}{1+x}$$

and

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$ **2**

(ii) Hence, show that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

and

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$ **2**

(iii) By letting $x = \frac{1}{2m+1}$

(α) Show that $\log\left(\frac{1+x}{1-x}\right) = \log\left(\frac{m+1}{m}\right)$ **1**

(β) Show that

$$\log\left(\frac{m+1}{m}\right) = 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \ldots\right)$$ **1**

(iv) Use a result from (i), (ii) or (iii) to find a simple fraction which approximates the value of $\log_e (1.001)$ correctly to 9 decimal places. **1**

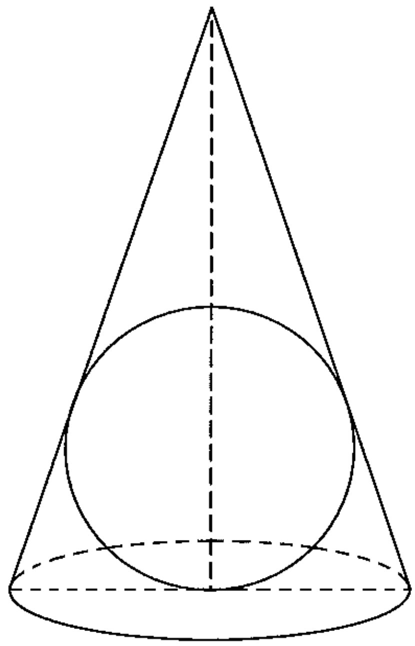
End of Question 7

Marks

Question 8 (15 marks) **[START A NEW PAGE]**

(a) You are trying to find the dimensions of the right circular cone of minimum volume which can be circumscribed about a **sphere of radius 20cm**, as shown below.

Let x cm = the radius of the base of the cone and let $(y + 20)$ cm = the altitude of the cone.



(i) Prove that $x^2 = \dfrac{400(y+20)}{y-20}$ using similar triangles. **2**

(ii) Hence, find the dimensions of the cone which make its volume a minimum. **3**

Question 8 continues on page 13

Marks

Question 8 (continued)

(b) By using the formula for $\tan(\alpha - \beta)$ in terms of $\tan\alpha$ and $\tan\beta$, answer the following questions.

(i) If $2x + y = \dfrac{\pi}{4}$, show that **2**

$$\tan y = \frac{1 - 2\tan x - \tan^2 x}{1 + 2\tan x - \tan^2 x}$$

(ii) Hence deduce that $\tan\dfrac{\pi}{8}$ is a root of the equation $t^2 + 2t - 1 = 0$ and find the exact value of $\tan\left(\dfrac{\pi}{8}\right)$ **3**

(c) For the series $S(x) = 1 + 2x + 3x^2 + \ldots + (n+1)x^n$,

find $(1-x)\,S(x)$ and hence find $S(x)$ **3**

(d) Find $\int_{-1}^{1} x^2 \sin^7 x \, dx$, giving reasons. **2**

End of Question 8

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension 2 Trial HSC 2011

1 (a) (i) $-\int \frac{dx}{x^2+2x-3} = -\int \frac{dx}{(x+3)(x-1)}$

Let $\frac{1}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$

$\therefore 1 \equiv A(x-1) + B(x+3)$

Let $x = 1$ $\therefore$ $1 = 4B$ $\therefore B = \frac{1}{4}$

Let $x = -3$ $\therefore 1 = -4A$ $\therefore A = -\frac{1}{4}$

$\therefore$ Answer: $-\int \frac{\frac{1}{4}}{x-1} - \frac{\frac{1}{4}}{x+3}\,dx$

$= -\frac{1}{4}\int \frac{1}{x-1} - \frac{1}{x+3}\,dx$

$= -\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$

(ii) $-\int \frac{2+x}{x^2+2x-3}\,dx$

$= -\frac{1}{2}\ln\left|\frac{x-1}{x+3}\right| - \int \frac{x}{x^2+2x-3}\,dx$

$= -\frac{1}{2}\ln\left|\frac{x-1}{x+3}\right| - \int \frac{x+1-1}{x^2+2x-3}\,dx$

$= -\frac{1}{2}\ln\left|\frac{x-1}{x+3}\right| - \frac{1}{2}\int \frac{2x+2}{x^2+2x-3}\,dx + \int \frac{1}{x^2+2x-3}\,dx$

$= -\frac{1}{2}\ln\left|\frac{x-1}{x+3}\right| - \frac{1}{2}\ln|x^2+2x-3| + \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$

$= -\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| - \frac{1}{2}\ln|x^2+2x-3| + C$

(b) (i) $\int \frac{dx}{\sqrt{-(x^2+2x-3)}}$

$= \int \frac{dx}{\sqrt{-((x+1)^2-4)}}$

$= \int \frac{dx}{\sqrt{4-(x+1)^2}}$

$= \sin^{-1}\left(\frac{x+1}{2}\right) + C$

(ii) $\sin^{-1}\left(\frac{x+1}{2}\right) + \int \frac{2x\,dx}{\sqrt{3-2x-x^2}}$

$= \sin^{-1}\left(\frac{x+1}{2}\right) + \int 2x\,(3-2x-x^2)^{-\frac{1}{2}}\,dx$

$= \sin^{-1}\left(\frac{x+1}{2}\right) + \int \frac{2x+2-2}{\sqrt{3-2x-x^2}}\,dx$

$= \sin^{-1}\left(\frac{x+1}{2}\right) - 2\int \frac{1}{\sqrt{3-2x-x^2}}\,dx - \int \frac{-2x-2}{\sqrt{3-2x-x^2}}\,dx$

$= \sin^{-1}\left(\frac{x+1}{2}\right) - 2\sin^{-1}\left(\frac{x+1}{2}\right) - \int (-2x-2)(3-2x-x^2)^{-\frac{1}{2}}\,dx$

$= -\sin^{-1}\left(\frac{x+1}{2}\right) - 2\sqrt{3-2x-x^2} + C$

(c) $\int 1\cdot\sqrt{x^2+a^2}\,dx$

$= x\sqrt{x^2+a^2} - \int x \times \frac{1}{2}\frac{2x}{\sqrt{x^2+a^2}}\,dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}\,dx$

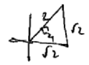
$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + \int \frac{a^2\,dx}{\sqrt{x^2+a^2}}$

$\therefore 2\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)$

$\therefore \int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)}{2} + C$

2 (a) (i)



$z^3 = 2\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$

$\therefore z = \sqrt[3]{2}\,\text{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)$

$z = \sqrt[3]{2}\,\text{cis}\left(\frac{\pi(1+8k)}{12}\right)$

(ii) For $z^3 - (\sqrt{2}+\sqrt{2}i) = 0$,

sum of roots $= 0$

$\therefore \sqrt[3]{2}\left(\text{cis}\frac{\pi}{12} + \text{cis}\frac{9\pi}{12} + \text{cis}\frac{-7\pi}{12}\right) = 0$

$\therefore \text{cis}\frac{\pi}{12} + \text{cis}\frac{9\pi}{12} + \text{cis}\frac{-7\pi}{12} = 0$

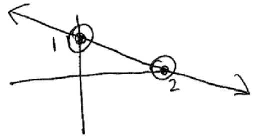
$\therefore$ real sum & imaginary sum $= 0$

$\therefore \cos\frac{\pi}{12} + \cos\frac{9\pi}{12} + \cos\frac{-7\pi}{12} = 0$

(b) (i) If $\frac{z-i}{z-2}$ is purely real,

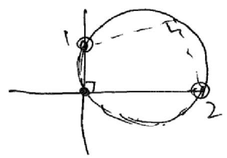
$\arg\left(\frac{z-i}{z-2}\right) = 0$ or $\pm\pi$

$\therefore \arg(z-i) - \arg(z-2) = 0$ or $\pm\pi$.



(ii) $\arg\left(\frac{z-i}{z-2}\right) = \pm\frac{\pi}{2}$

$\therefore \arg(z-i) - \arg(z-2) = \pm\frac{\pi}{2}$



(c) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$\therefore \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta = \cos 5\theta + i\sin 5\theta$

Equating reals,

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)(1-\cos^2\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) $16x^5 - 20x^3 + 5x = \frac{1}{2}$

Let $x = \cos\theta$

$\therefore$ Solve $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \frac{1}{2}$

$\therefore \cos 5\theta = \frac{1}{2}$

$5\theta = \pm\frac{\pi}{3} + 2k\pi$

$\theta = \pm\frac{\pi}{15} + \frac{2k\pi}{5}$

$\therefore x = \cos\left(\frac{2k\pi}{5} \pm \frac{\pi}{15}\right)$

Since 5 solns:

$x = \cos\frac{\pi}{15}, \cos\frac{\pi}{3}, \cos\frac{7\pi}{15}, \cos\frac{11\pi}{15}, \cos\frac{13\pi}{15}$

$= \cos\frac{\pi}{15}, \frac{1}{2}, \cos\frac{7\pi}{15}, \cos\frac{11\pi}{15}, \cos\frac{13\pi}{15}$

(iii) Product of roots of $32x^5 - 40x^3 + 10x - 1 = 0$

$\frac{1}{2}\cos\frac{\pi}{15}\cos\frac{7\pi}{15}\cos\frac{11\pi}{15}\cos\frac{13\pi}{15}$

Note: $32x^5 - 40x^3 + 10x - 1 = 0$ has product of roots $= -\left(\frac{-1}{32}\right) = \frac{1}{32}$

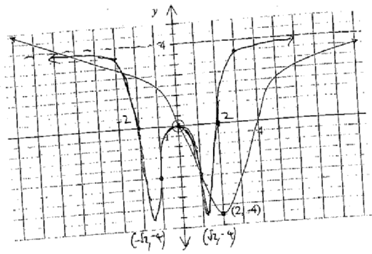
$\therefore \frac{1}{2}\cos\frac{\pi}{15}\cos\frac{7\pi}{15}\cos\frac{11\pi}{15}\cos\frac{13\pi}{15} = \frac{1}{32}$

$\therefore \cos\frac{\pi}{15}\cos\frac{7\pi}{15}\cos\frac{11\pi}{15}\cos\frac{13\pi}{15} = \frac{1}{16}$

3 (a)

(i) $y = f(x^2)$

(ii) see below



(b) $4x + 2\left(y + x\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$

$4x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2x + 6y) = -(4x + 2y)$

$\therefore \frac{dy}{dx} = -\frac{4x+2y}{2x+6y} = -\left(\frac{2x+y}{x+3y}\right)$

For vertical tangents, $x + 3y = 0$

$\therefore x = -3y$

$\therefore 2(-3y)^2 + 2(-3y)y + 3y^2 = 15$

$18y^2 - 6y^2 + 3y^2 = 15$

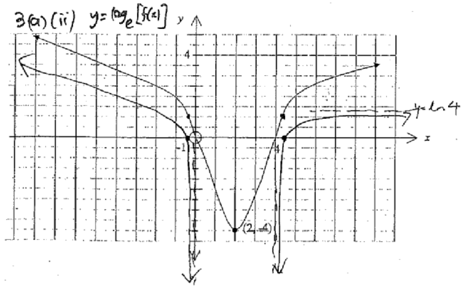
$15y^2 = 15 \quad \therefore y = \pm 1$

$y = 1 \rightarrow x = -3$

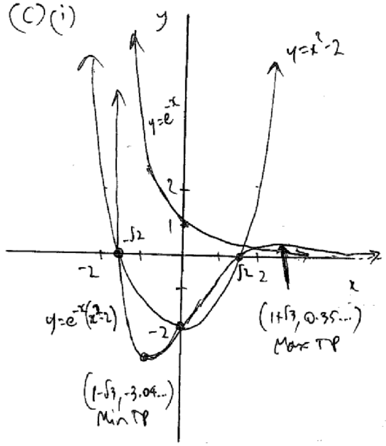
$y = -1 \rightarrow x = 3$

$\therefore$ x coordinates are ± 3

3(a)(ii) $y = \log_e[f(x)]$



(C)(i)



(ii) $\frac{dy}{dx} = -e^{-x}(x^2-2) + 2xe^{-x}$

$= e^{-x}\left[2x - x^2 + 2\right]$

$= e^{-x} \cdot -\left[x^2 - 2x - 2\right]$

$= -e^{-x}\left[(x-1)^2 - 3\right] = 0$ for stat pts

$\therefore (x-1)^2 = 3$

$x - 1 = \pm\sqrt{3}$

$x = 1 \pm \sqrt{3}$

$\left[x \doteq \begin{matrix} 2.732... \\ -0.732... \end{matrix}\right]$

(4) (a) $\frac{x^2}{8} + \frac{y^2}{4} = 1$

(i) $a = 2\sqrt{2}$

$b = 2$

$b^2 = a^2(1 - e^2)$

$4 = 8(1 - e^2)$

$1 - e^2 = \frac{1}{2}$

$e^2 = \frac{1}{2}$

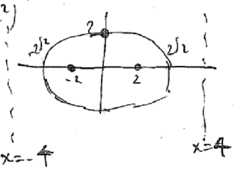
$e = \frac{1}{\sqrt{2}}$

$ae = 2$

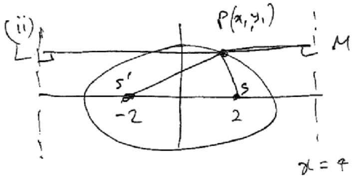
$\frac{a}{e} = 4$

$\therefore$ Foci $(\pm 2, 0)$

Directrices $x = \pm 4$



(ii)



$x = -4$

Construct LPM straight horizontal line

Now $\frac{PS}{PM} = \frac{1}{\sqrt{2}}$ & $\frac{PS'}{PL} = \frac{1}{\sqrt{2}}$

$\therefore PM = \sqrt{2}PS$ & $PL = \sqrt{2}PS'$

and $PM + PL = 8$

$\therefore 8 = \sqrt{2}(PS + PS')$

$\therefore PS + PS' = \frac{8}{\sqrt{2}} = 4\sqrt{2}$ which is independent of P's position

(iii) $\frac{x}{4} + \frac{y}{2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-\frac{x}{4}}{\frac{y}{2}} = \frac{-x}{2y} \quad \therefore m = \frac{-x_1}{2y_1}$

$\therefore y - y_1 = \frac{-x_1}{2y_1}(x - x_1)$

$\therefore 2yy_1 - 2y_1^2 = x_1^2 - xx_1$

$xx_1 + 2yy_1 = x_1^2 + 2y_1^2$

$\therefore \frac{xx_1}{8} + \frac{yy_1}{4} = \frac{x_1^2}{8} + \frac{y_1^2}{4}$

Now $\frac{x_1^2}{8} + \frac{y_1^2}{4} = 1$ from eqn of ellipse

$\therefore \frac{xx_1}{8} + \frac{yy_1}{4} = 1$

$\therefore xx_1 + 2yy_1 = 8$

(iv) For T:

$4x_1 + 2yy_1 = 8$

$2yy_1 = 8 - 4x_1$

$y = \frac{8 - 4x_1}{2y_1} = \frac{4 - 2x_1}{y_1}$

$\therefore T\left(4, \frac{4 - 2x_1}{y_1}\right)$

$M_{PS} = \frac{y_1 - 0}{x_1 - 2} = \frac{y_1}{x_1 - 2}$

$M_{TS} = \frac{\frac{4-2x_1}{y_1} - 0}{4 - 2} = \frac{4 - 2x_1}{2y_1}$

$M_{PS} \times M_{TS} = \frac{y_1}{x_1 - 2} \times \frac{4 - 2x_1}{2y_1}$

$= \frac{y_1}{2y_1} \times \frac{2(2 - x_1)}{x_1 - 2}$

$= \frac{-(x_1 - 2)}{x_1 - 2} = -1$

$\therefore \angle PST$ is a right angle

(b)

$y = \frac{c^2}{x} \quad \therefore \frac{dy}{dx} = \frac{-c^2}{x^2}$

$\therefore M_{normal} = \frac{(ct)^2}{c^2} = t^2$

$\therefore y - \frac{c}{t} = t^2(x - ct)$ is normal eqn

For R: solve simultaneously with $y = x$

$\therefore x - \frac{c}{t} = t^2(x - ct)$

$tx - c = t^3(x - ct)$

$x(t - t^3) = c - ct^4 = c(1 - t^4)$

$x = \frac{c(1 - t^4)}{t(1 - t^2)} = \frac{c}{t}(1 + t^2)$

$\therefore R\left(\frac{c}{t}(1 + t^2), \frac{c}{t}(1 + t^2)\right)$

$R\left(\frac{c}{t} + ct, \frac{c}{t} + ct\right)$

= sum of ordinates of T !!

5 (a) (i) $p(\beta)=0$

$\therefore a\beta^3+b\beta^2+c\beta+d=0$

$\therefore a\beta^3+b\beta^2+c\beta=-d$

$\div\beta \quad \therefore a\beta^2+b\beta+c=\frac{-d}{\beta}$

Note: if $\beta=0$, that means $d=0$, which still satisfies β dividing d

Since a,b,c & β are integers, LHS is an integer

$\therefore \frac{-d}{\beta}$ is an integer

Since d & β are integers, β must divide exactly into d

(ii) Need to test integers $\pm1, \pm3$

$q(1)=2 \qquad q(3)=30$

$q(-1)=-18 \qquad q(-3)=-126$

$\therefore \pm1, \pm3$ not roots

Since they are only factors of 3, no integer root

(b) (i) $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{(\alpha+\beta+\gamma)^2-(\alpha^2+\beta^2+\gamma^2)}{2}$

$=\frac{0^2-(-2)}{2}=1$

$\alpha\beta\gamma=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}=\frac{1}{-\frac{1}{10}}=-10$

(ii) Let $a=1$

$\therefore c=1 \quad (\alpha\beta+\beta\gamma+\alpha\gamma)$

$d=10 \quad (-\alpha\beta\gamma)$

$b=0 \quad (-(\alpha+\beta+\gamma))$

$\therefore x^3+x+10=0$

(c)(i) $\alpha\beta\gamma=4$

(ii)(A) Let $y=x^2 \quad \therefore x=\sqrt{y}$

$\therefore y\sqrt{y}+y+2\sqrt{y}-4=0$

$\sqrt{y}(y+2)=4-y$

$y(y^2+4y+4)=16-8y+y^2$

$y^3+3y^2+12y-16=0$

$\therefore x^3+3x^2+12x-16=0$

(B) Roots: $\alpha(\alpha\beta\gamma)$, $\beta(\alpha\beta\gamma)$, $\gamma(\alpha\beta\gamma)$

$=4\alpha, 4\beta, 4\gamma$

$\therefore$ Let $y=4x \quad \therefore x=\frac{y}{4}$

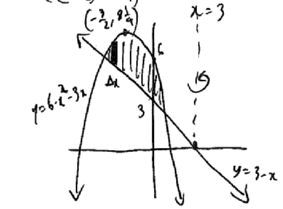
$\therefore \frac{y^3}{64}+\frac{y^2}{16}+\frac{y}{2}-4=0$

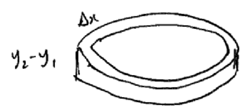
$y^3+4y^2+32y-256=0$

$\therefore x^3+4x^2+32x-256=0$

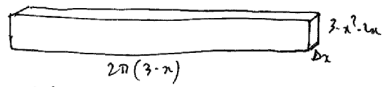
(6) $y=-(x^2+3x-6)$

$y=-\left((x+\frac{3}{2})^2-8\frac{1}{4}\right)=-(x+\frac{3}{2})^2+8\frac{1}{4}$





Now $y_2-y_1=$
$6-x^2-3x-(3-x)$
$=3-x^2-2x$



Points of intersection: $3-x^2-2x=0$

$\therefore x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ or 1

$\therefore V=2\pi\int_{-3}^{1}(3-x)(3-x^2-2x)\,dx$

$=2\pi\int_{-3}^{1}9-3x^2-6x-3x+x^3+2x^2\,dx$

$=2\pi\int_{-3}^{1}9-x^2-9x+x^3\,dx=2\pi\left[9x-\frac{x^3}{3}-\frac{9x^2}{2}+\frac{x^4}{4}\right]_{-3}^{1}$

$=2\pi\left(4\frac{5}{12}-\,-38\frac{1}{4}\right)=\frac{256\pi}{3}$

(b) (i) $\cos(n+2)x+\cos nx=$

$\cos(n+1)x\cos x-\sin(n+1)x\sin x$

$+$

$\cos(n+1)x\cos x+\sin(n+1)x\sin x$

$=2\cos(n+1)x\cos x$

(ii) $U_{n+2}+U_n-2U_{n+1}=$

$\int_0^{\pi}\frac{1-\cos(n+2)x+1-\cos nx-2(1-\cos(n+1)x)}{1-\cos x}\,dx$

$=\int_0^{\pi}\frac{2-\cos(n+2)x-\cos nx-2+2\cos(n+1)x}{1-\cos x}\,dx$

$=\int_0^{\pi}\frac{2\cos(n+1)x-\cos(n+2)x-\cos nx}{1-\cos x}\,dx$

$=\int_0^{\pi}\frac{2\cos(n+1)x-(2\cos(n+1)x\cos x)}{1-\cos x}\,dx$

$=\int_0^{\pi}\frac{2\cos(n+1)x\left[1-\cos x\right]}{1-\cos x}\,dx$

$=\int_0^{\pi}2\cos(n+1)x\,dx$

$=\left[\frac{2\sin(n+1)x}{n+1}\right]_0^{\pi}$

$=\frac{2}{n+1}\left[\sin(n+1)\pi-\sin(n+1)0\right]$

$=\frac{2}{n+1}\left[0-0\right]$ since $\sin k\pi=0$ for k integer

$=0$

(iii) $U_0=\int_0^{\pi}0\,dx=0$

$U_1=\int_0^{\pi}1\,dx=\left[x\right]_0^{\pi}=\pi$

$U_2+U_0-2U_1=0$

$\therefore U_2=2\pi$

$U_3+U_1-2U_2=0$

$\therefore U_3=3\pi$

(iv)

To change limits, let's try this substitution.

Let $x=2\theta$. $\quad dx=2d\theta$

$\therefore$ when $\theta=0$, $x=0$

when $\theta=\frac{\pi}{2}$, $x=\pi$

$\therefore \frac{1}{2}\int_0^{\pi}\frac{\sin^2 3\theta}{\sin^2\theta}\,d\theta$

Now if $x=2\theta$,

$\cos x=\cos 2\theta=2\cos^2\theta-1$

$=1-2\sin^2\theta$

$\therefore \sin^2\theta=\frac{1-\cos x}{2}$

& $\cos 3x=\cos 6\theta=1-2\sin^2 3\theta$

$\therefore \sin^2 3\theta=\frac{1-\cos 3x}{2}$

$\therefore \frac{1}{2}\int_0^{\pi}\frac{\frac{1-\cos 3x}{2}}{\frac{1-\cos x}{2}}\,dx=\frac{1}{2}\int_0^{\pi}\frac{1-\cos 3x}{1-\cos x}\,dx$

$=\frac{1}{2}U_3=\frac{3\pi}{2}$

8(c) $(1-x)S(x)=1+2x+3x^2+4x^3+\ldots+(n+1)x^n$

$-x-2x^2-3x^3-4x^4-\ldots-nx^n-(n+1)x^{n+1}$

$=\underbrace{1+x+x^2+x^3+\ldots+x^n}-(n+1)x^{n+1}$

GP: $n+1$ terms, $a=1$, $r=x$

$\therefore S(x)=\frac{\frac{1(x^{n+1}-1)}{x-1}-(n+1)x^{n+1}}{1-x}$

$=\frac{x^{n+1}-1-(n+1)x^{n+1}(x-1)}{(x-1)(1-x)}$

$=\frac{x^{n+1}-1-(n+1)x^{n+2}+(n+1)x^{n+1}}{-(x-1)^2}$

$=\frac{(n+2)x^{n+1}-(n+1)x^{n+2}-1}{-(x-1)^2}=\frac{1+(n+1)x^{n+2}-(n+2)x^{n+1}}{(x-1)^2}$

(7)(i) $\frac{1}{2}v^2 = \int -kx^{-2}\,dx$

$\frac{1}{2}v^2 = \frac{k}{x} + c$

When $x = R$, $v = u$

$\therefore \frac{1}{2}u^2 = \frac{k}{R} + c$

$\therefore c = \frac{u^2}{2} - \frac{k}{R}$

$\therefore \frac{1}{2}v^2 = \frac{k}{x} + \frac{u^2}{2} - \frac{k}{R}$

$\therefore v^2 = \frac{2k}{x} + u^2 - \frac{2k}{R}$

$v^2 = u^2 + 2k\left(\frac{1}{x} - \frac{1}{R}\right)$

Now to find k:

When $x = R$, $\ddot{x} = -g$

$\therefore -g = \frac{-k}{R^2}$ $\therefore k = gR^2$

$\therefore v^2 = u^2 + 2(gR^2)\left(\frac{1}{x} - \frac{1}{R}\right)$

$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

(ii) Greatest height when $v = 0$

$\therefore u^2 = 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

$\frac{u^2}{2gR^2} = \frac{1}{R} - \frac{1}{x}$

$\frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{2gR - u^2}{2gR^2}$

$\therefore x = \frac{2gR^2}{2gR - u^2}$

But x is distance from centre of earth.

$\therefore H = \frac{2gR^2}{2gR - u^2} - R = \frac{2gR^2 - 2gR^2 + Ru^2}{2gR - u^2}$

$= \frac{u^2R}{2gR - u^2}$

(iii) To escape, $H \to \infty$

ie $2gR \to u^2$

$\therefore .002 \times 6400 \to u^2$ [units in km & seconds]

$128 \to u^2$

$\therefore u \to \sqrt{128} \doteqdot 11.3\ldots$ km/sec

$\therefore$ If particle exceeds 12km/sec, will escape!

(b)(i) $1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$ is an infinite series, $a = 1$, $r = -x$. Note $|r| < 1$ since $0 < x < 1$

$\therefore$ The sum $= \frac{1}{1 - -x} = \frac{1}{1+x}$

* Integrating both sides:

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \ln(1+x) + c$

when $x = 0$, $0 = 0 + c$ $\therefore c = 0$

$\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \ln(1+x)$

(ii) Replace x by $-x$

$\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

$\therefore \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

$- \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)$

$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$

$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

(iii)(α) $\ln\left(\frac{1 + \frac{1}{2m+1}}{1 - \frac{1}{2m+1}}\right) = \ln\left(\frac{2m+1+1}{2m+1-1}\right) = \ln\left(\frac{2m+2}{2m}\right)$

$= \ln\left(\frac{m+1}{m}\right)$

(β) $2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) = 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \ldots\right)$

$\therefore$ Proven

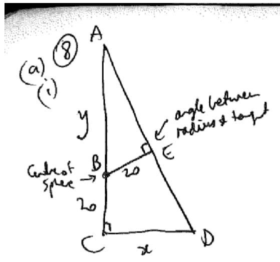
(iv) let $m = 1000$ $\therefore \log\left(1\frac{1}{1000}\right) =$

$2\left(\frac{1}{2001} + \frac{1}{3(2001)^3} + \frac{1}{5(2001)^5} + \frac{1}{7(2001)^7} + \ldots\right)$

$= 9.995003331 \times 10^{-4}$ ↗ To 9dp, $\frac{2}{2001}$ does the trick!

$= 0.000999500$

(a) (8) (i)



$\Delta ABE \parallel\!\mid \Delta ADC$ (equiangular)

since $\angle A$ is common & $\angle AEB = \angle ACD = 90°$

$\therefore \frac{20}{x} = \frac{y}{\sqrt{x^2 + (y+20)^2}}$

$400(x^2 + (y+20)^2) = x^2y^2$

$400x^2 + 400y^2 + 16000y + 160000 = x^2y^2$

$x^2(y^2 - 400) = 400y^2 + 16000y + 160000$

$x^2(y^2 - 400) = 400(y^2 + 40y + 400)$

$x^2(y-20)(y+20) = 400(y+20)^2$

$x^2 = \frac{400(y+20)}{y-20}$ [$\div$ by $y+20$ which $\neq 0$]

(ii)

$V_{cone} = \frac{1}{3}\pi x^2(y+20)$

$V = \frac{1}{3}\pi \frac{400(y+20)(y+20)}{y-20}$

$V = \frac{400\pi}{3}\frac{(y+20)^2}{y-20}$

$\frac{dV}{dy} = \frac{400\pi}{3}\left[\frac{2(y+20)(y-20) - (y+20)^2}{(y-20)^2}\right]$

$= \frac{400\pi}{3}\left[\frac{(y+20)[2y-40-y-20]}{(y-20)^2}\right]$

$= \frac{400\pi}{3}\left[\frac{(y+20)(y-60)}{(y-20)^2}\right]$

For stat pts, $(y+20)(y-60) = 0$

$\therefore y = \cancel{-20}$ or 60.

cone has no height

Check for minimum:

y	50	60	70
$\frac{dV}{dy}$	$\frac{400\pi}{3}\times\frac{70\times-10}{30^2}$	0	$\frac{400\pi}{3}\times\frac{90\times10}{50^2}$

$\therefore$ Min Volume

$\therefore$ dimensions: Height = 80 cm

$x^2 = \frac{400 \times 80}{40} = 800$ $\therefore$ Radius $= \sqrt{800}$ cm $= 20\sqrt{2}$ cm

(b)(i) $y = \frac{\pi}{4} - 2x$

$\therefore \tan y = \frac{1 - \tan 2x}{1 + \tan 2x} = \frac{1 - \frac{2\tan x}{1 - \tan^2 x}}{1 + \frac{2\tan x}{1 - \tan^2 x}}$

$= \frac{1 - \tan^2 x - 2\tan x}{1 - \tan^2 x + 2\tan x}$

(ii)

Let $x = \frac{\pi}{8}$ $\therefore y = 0$.

$\therefore \tan 0 = 0 = \frac{1 - 2\tan\frac{\pi}{8} - \tan^2\frac{\pi}{8}}{1 + 2\tan\frac{\pi}{8} - \tan^2\frac{\pi}{8}}$

$\therefore 1 - 2\tan\frac{\pi}{8} - \tan^2\frac{\pi}{8} = 0$

$\therefore \tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

$\therefore$ For $k^2 + 2k - 1 = 0$, $k = \tan\frac{\pi}{8}$ is a soln

$\therefore \tan\frac{\pi}{8} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

since $\tan\frac{\pi}{8}$ is +ve, $\tan\frac{\pi}{8} = \sqrt{2} - 1$

(c) Previous page

(d) Let $f(x) = x^2\sin^7 x$

$\therefore f(-x) = (-x)^2\sin^7(-x)$

$= x^2(-\sin x)^7$

$= x^2(-1)\sin^7 x$

$= -x^2\sin^7 x$

$= -f(x)$

$\therefore f(-x) = -f(x)$ $\therefore$ Odd function

$\therefore \int_{-1}^{1} x^2\sin^7 x\,dx = 0$

↑ since symmetrical limits about y-axis