BAULKHAM HILLS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 120
- Attempt Questions 1 – 8
- All questions are of equal value
- Start a separate piece of paper for each question.
- Put your student number and the question number at the top of each sheet.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

<u>Question 1</u> ***(15 marks)*** Use a *separate* piece of paper — ***Marks***

a) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$ — *2*

b) Use integration by parts to evaluate $\displaystyle\int_0^1 x\tan^{-1} x\,dx$ — *3*

c) (i) Express $\dfrac{10+x-x^2}{(x+1)(x^2+3)}$ in the form $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+3}$ — *3*

(ii) Hence find $\displaystyle\int \frac{10+x-x^2}{(x+1)(x^2+3)}dx$ — *2*

d) Find $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$ — *2*

e) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\,dx$ — *3*

<u>Question 2</u> ***(15 marks)*** Use a *separate* piece of paper

a) Let $z=1+2i$ and $\omega=3+i$. Find $\dfrac{1}{z\omega}$ in the form $x+iy$. — *2*

b) Find the real numbers *a* and *b* such that $(a+bi)^2=9+40i$ — *3*

c) (i) Determine the modulus and argument of $-1+i$ — *2*

(ii) Hence find the least positive integer value of *n* for which $(-1+i)^n$ is real. — *1*

d) If $z=x+iy$, describe the locus of *z* if $2|z|=z+\overline{z}+4$ — *2*

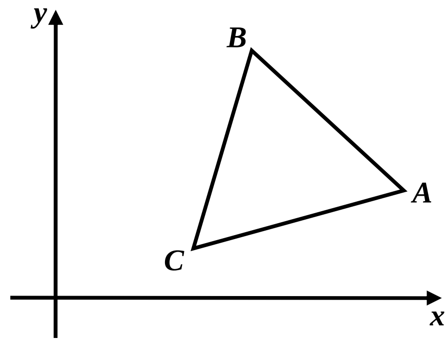
e) On an Argand diagram, sketch the region specified by both the conditions — *3*

$$|z+3-4i|\le 5 \qquad \text{and} \qquad \mathrm{Re}(z)\le 1$$

You must show the intercepts with the axes, but you do not need to find other points of intersection.

Question 2 ***(continued)*** ***Marks***

f) *2*

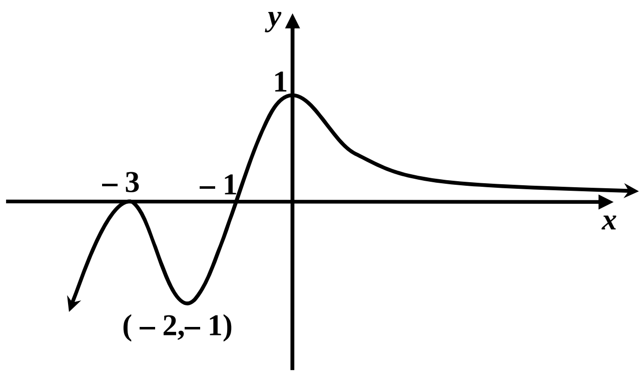


The points A, B and C in the Argand diagram are the vertices of an equilateral triangle described anticlockwise. Show that the complex numbers a, b and c representing A, B and C satisfy;

$$2c = (a+b) + i\sqrt{3}(b-a)$$

<u>Question 3</u> ***(15 marks)*** Use a *separate* piece of paper

a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following;

(i) $y = f(|x|)$ *1*

(ii) $y = f(1-x)$ *1*

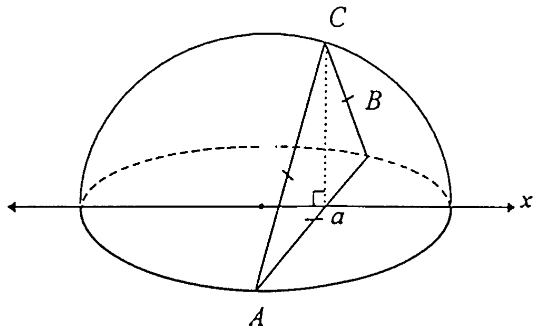
(iii) $y = \dfrac{1}{f(x)}$ *2*

(iv) $y = \sqrt{f(x)}$ *2*

(v) $y = \ln f(x)$ *2*

Question 3*(continued)* ***Marks***

b)



The solid shape above has a circular base of radius 3 units in the horizontal plane. Vertical cross-sections perpendicular to the diameter along the x axis are equilateral triangles as shown in the diagram.

(i) A vertical slice of width Δx is positioned as the point $x = a$. *2*
If its volume is denoted by ΔV , show that $\Delta V = \sqrt{3}\left(9 - x^2\right)\Delta x$

(ii) Hence determine the volume of the solid. *2*

c) If ω is the root of $\omega^5 - 1 = 0$ with the smallest positive argument, find the quadratic equation with roots $\omega + \omega^4$ and $\omega^2 + \omega^3$ *3*

<u>Question 4</u> ***(15 marks)*** Use a *separate* piece of paper

a) Find all roots of the equation $2x^3 - 5x^2 - 4x + 12 = 0$ given that two of the roots are equal. *3*

b) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.

(i) Show that the tangent to the ellipse at the point $P(a\cos\theta, b\sin\theta)$ has the equation $bx\cos\theta + ay\sin\theta - ab = 0$. *2*

(ii) Prove $b^2\cos^2\theta + a^2\sin^2\theta = a^2\left(1 - e^2\cos^2\theta\right)$, where e is the eccentricity. *2*

(iii) R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at P. Show that $SR \cdot S'R' = b^2$. *3*

c) (i) Prove that $\tan^{-1} n - \tan^{-1}(n-1) = \tan^{-1}\dfrac{1}{n^2 - n + 1}$, where n is a positive integer. *2*

(ii) Hence evaluate $\tan^{-1} 1 + \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} + \ldots + \tan^{-1}\dfrac{1}{n^2 - n + 1}$ *2*

(iii) Hence find $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{n^2 - n + 1}$ *1*

<u>Question 5</u> ***(15 marks)*** Use a *separate* piece of paper **_Marks_**

a) The zeros of $x^3 - 3x^2 - 2x + 4$ are α, β and γ

(i) Find a cubic polynomial whose zeros are α^2, β^2 and γ^2 *2*

(ii) Hence, or otherwise, find the value of $\alpha^2 + \beta^2 + \gamma^2$ *1*

(iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ *2*

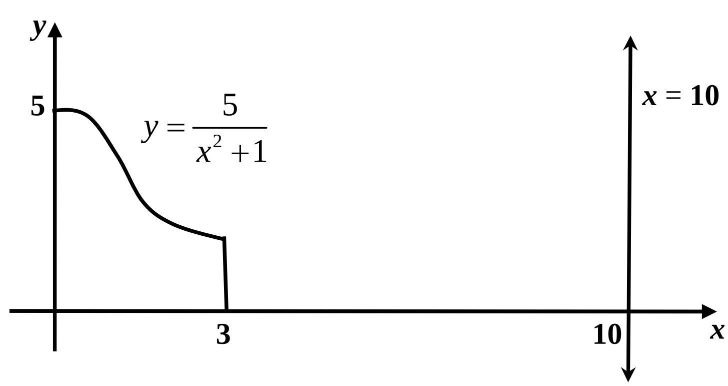
b) A particle of unit mass is thrown vertically downwards with an initial velocity of u. It experiences a resistive force of magnitude kv^2 where v is its velocity.

Let V be the terminal velocity of the particle.

(i) Show that $V = \sqrt{\dfrac{g}{k}}$, where g is the acceleration due to gravity. *2*

(ii) Show that $v^2 = V^2 + \left(u^2 - V^2\right)e^{-2kx}$. *3*

c) A circular flange is formed by rotating the region bounded by the curve $y = \dfrac{5}{x^2+1}$, the x axis and the lines $x = 0$ and $x = 3$ through one complete revolution about the line $x = 10$. All measurements are in centimetres.



(i) Use the method of cylindrical shells to show the volume V cm^3 of the flange is given by $V = 10\pi \int_0^3 \dfrac{10-x}{x^2+1}\,dx$ *3*

(ii) Hence find the volume of the flange to the nearest cm^3. *2*

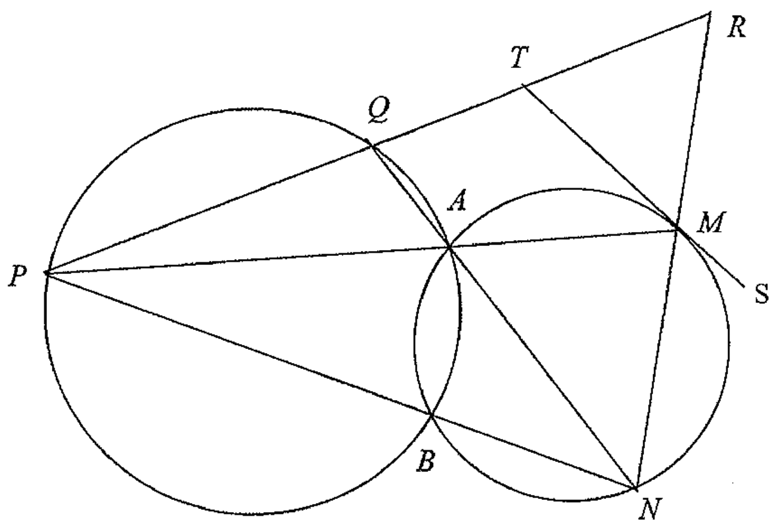
Question 6 ***(15 marks)*** Use a *separate* piece of paper **Marks**

a) (i) Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has the equation $x + t^2 y = 2ct$. *2*

(ii) The tangents to the rectangular hyperbola $xy = c^2$ at the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, intersect at *R*. Show that *R* has the coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. *3*

(iii) It is known that *P* and *Q* are variable points on the hyperbola which move so that $pq = 1$. Find the locus of *R* and state any restrictions on the values of *x* for this locus. *2*

b)



In the diagram, the two circles intersect at *A* and *B*. *P* is a point on one circle. *PA* and *PB* produced meet the other circle at *M* and *N* respectively. *NA* produced meets the first circle at *Q*. *PQ* and *NM* produced meet at *R*. The tangent at *M* to the second circle meets *PR* at *T*.

(Note: on page 10 there is a copy of the above diagram that you can use, please detach and include with your solutions)

(i) Show that ***QAMR*** is a cyclic quadrilateral *3*

(ii) Show that *TM* = *TR* *3*

c) The continued surd $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\ldots\ldots}}}}} = L$ *2*

Find the exact value of *L*.

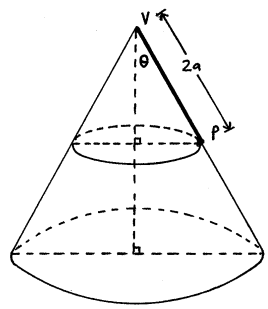
Question 7 ***(15 marks)*** Use a *separate* piece of paper **Marks**

a) The gradient $\frac{dy}{dx}$ of a curve at a point (x, y) satisfies $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

(i) By differentiating with respect to *x*, show that either $\frac{d^2y}{dx^2} = 0$ or $2\frac{dy}{dx} = x$ *2*

(ii) Hence show that the curve is either a straight line or a parabola. *2*

b) A circular cone of semi vertical angle θ is fixed with its vertex upwards as shown. A particle *P*, of mass 2*m* kg, is attached to the vertex *V* by a light inextensible string of length 2*a* metres.



The particle *P* rotates with uniform velocity ω radians/second in a horizontal circle on he outside surface of the cone and in contact with it.

(i) Show that the tension (*T*) in the sting is equal to $2m\left(g\cos\theta + r\omega^2\sin\theta\right)$ *2*

(ii) Find the normal force (*N*) on *P* sting is equal to $2m\left(g\sin\theta - r\omega^2\cos\theta\right)$ *2*

(iii) Show that, for the particle to remain in uniform circular motion on the surface of the cone, then $\omega < \sqrt{\frac{g}{2a\cos\theta}}$ where *g* is acceleration due to gravity. *2*

c) (i) Show that $\int_0^{\frac{\pi}{4}} \sec x dx = \ln\left(\sqrt{2}+1\right)$ *1*

(ii) Let $I_n = \int_0^{\frac{\pi}{4}} \sec^n \theta d\theta$, show that $I_n = \frac{1}{n-1}\left(\left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\right)$ for $n \geq 2$ *3*

(iii) Hence find I_3 *1*

<u>Question 8</u> ***(15 marks)*** Use a *separate* piece of paper **_Marks_**

a) A three digit number has a hundreds digit of *a*, a tens digit of *b* and a units digit of *c*. *2*

If $a + b + c$ is divisible by 3, show that the three digit number is divisible by 3.

b) (i) Use de Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ *1*

(ii) Hence solve $8x^3 - 6x - 1 = 0$ *2*

(iii) Deduce that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$ *2*

c) The Fibonacci sequence of numbers, $F_1, F_{2,\ldots,}$ is defined by $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

(i) Prove that $F_{2n+3}F_{2n+1} - F_{2n+2}^2 = -F_{2n+2}F_{2n} + F_{2n+1}^2$. *2*

(ii) Prove by induction that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$, for all positive integers. *3*

(iii) Hence deduce that $F_{2n}^2 + 1$ is divisible by F_{2n+1} *1*

(iv) Prove that $F_{2n-1}^2 + 1$ is divisible by F_{2n+1}. *2*

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log x, \quad x > 0$

Question 6 b)

Please detach and include with your solutions.