Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2007

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	3, 5	
Applies appropriate algebraic techniques to complex numbers and polynomials	2, 4	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	1, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \cos^5 x \sin x \; dx$. **2**

(b) Find $\int \frac{2x}{\sqrt{x^2-4}}\, dx$ using the substitution $u = x^2 - 4$. **3**

(c) (i) Express $\frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)}$ as a sum of partial fractions. **2**

(ii) Hence show that $\int_0^3 \frac{2-x^2}{\left(x^2+1\right)\left(x^2+4\right)} dx = \tan^{-1}\left(\frac{3}{11}\right)$. **4**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x}$. **4**

Question 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 1 + i$ and $w = 1 - 2i$. Find in the form $x + iy$,

(i) $z\overline{w}$ **1**

(ii) $3z + iw$ **1**

(iii) $\dfrac{w}{z}$ **1**

(b) Let $\beta = -1 + i$

(i) Express β in modulus-argument form. **2**

(ii) Express β^4 in modulus-argument form. **1**

(iii) Hence evaluate β^{20} **1**

(c) (i) Sketch, on the same Argand diagram, the locus specified by, **4**

1. $|z - 9| = |z + 1|$

2. $|z - 2 + i| = 2$

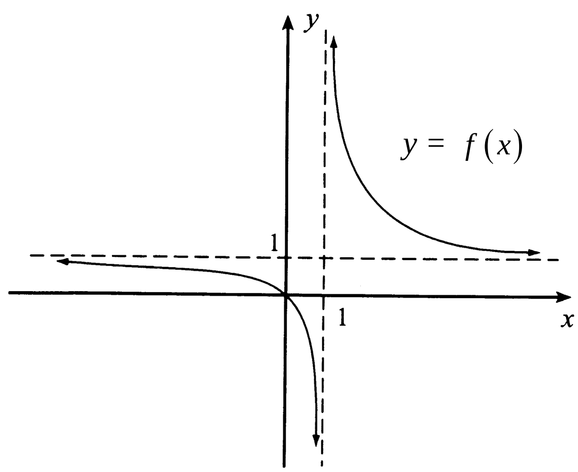
(ii) Hence write down all the values of z which satisfy simultaneously **1**

$$|z - 9| = |z + 1| \quad \text{and} \quad |z - 2 + i| = 2$$

(d) Prove $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and interpret this result geometrically. **3**

Question 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram bellows shows the graph of $y = f(x)$



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x-1)-1$ **2**

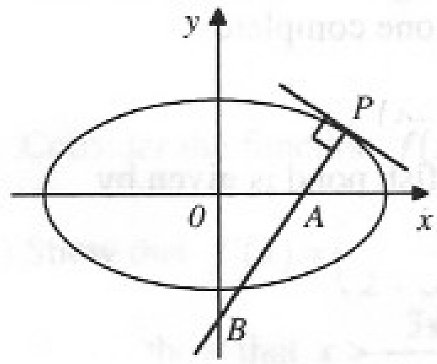
(ii) $y = |f(x)|$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = \log_e(f(x))$ **2**

(Question 3 continues over)

(b) $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \dfrac{\pi}{2}$, is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > b > 0$.



The normal at P cuts the x axis at A and the y axis at B.

(i) Show that the normal at P has the equation **2**

$$ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$$

(ii) Show that triangle OAB has areas $\dfrac{\left(a^2 - b^2\right)^2 \sin\theta\cos\theta}{2ab}$ **2**

(iii) Find the maximum area of the triangle OAB and the coordinates of P when this maximum occurs. **3**

Question 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that α, β and γ are the roots to the equation $x^3 - x^2 + 5x - 3 = 0$, find the equation whose roots are αβ, αγ and βγ **3**

(b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest possible argument.

Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$

(i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$ **2**

(ii) Show $\theta + \phi = -1$ and $\theta\phi = 2$ **3**

Hence write a quadratic equation whose roots are θ and ϕ

(iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$ **2**

(iv) Write α in modulus argument form and show **2**

$$\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2} \quad \text{and} \quad \sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

(c) The polynomial $P(z)$ is defined by $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$. **3**

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors.

Question 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the curve given by $5y - xy = x^2 - x - 2$

(i) Show that the curve has stationary points at $5 \pm 3\sqrt{2}$ **2**

(ii) Explain why the curve approaches that of $y = -x - 4$ as $x \to \pm\infty$ **2**

(b) For the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$, find

(i) The eccentricity. **1**

(ii) The coordinates of the foci. **1**

(iii) The equations of the directrices. **1**

(iv) The equations of the asymptotes. **1**

(v) Sketch the hyperbola indicating the foci, the directrices and the asymptotes. **1**

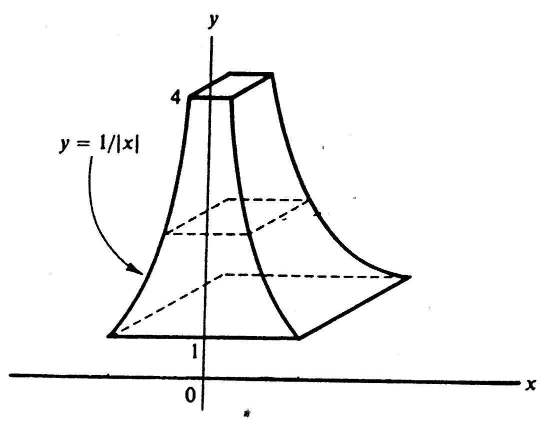
(vi) Show that the point $P\left(2\sec\theta, \sqrt{5}\tan\theta\right)$ lies on the hyperbola and prove that the tangent to the hyperbola at P has the equation **2**

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$$

(vii) If the tangent at P cuts the asymptotes at L and M, prove that $LP = PM$ and the area of triangle OLM is independent of the position of P. **4**

Question 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

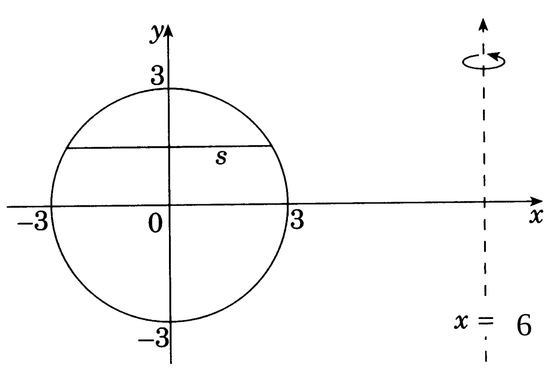
(a) The plan of a steeple is bounded by the curve $y = \dfrac{1}{|x|}$ and the lines $y = 4$ and $y = 1$.



Each horizontal cross-section is a square.

Find the volume of the steeple. **4**

(b) The circle $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a ring.

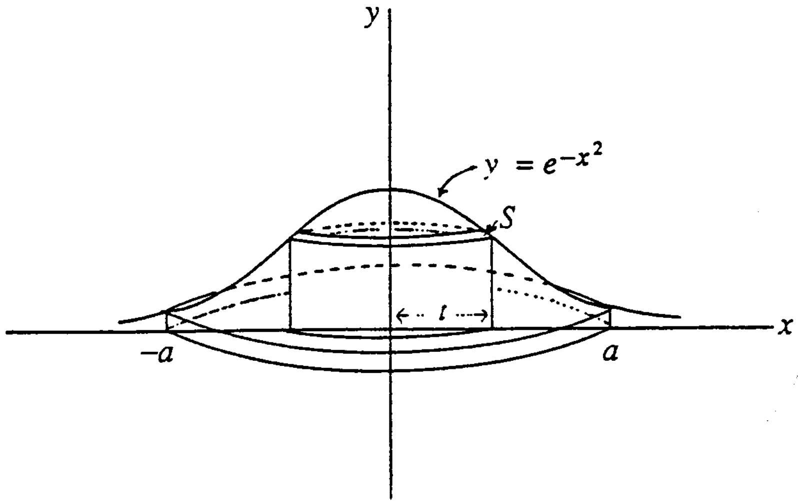


(i) When the circle is rotated, the line segment S at height y sweeps out an annulus. **2**

Find the area of the Annulus.

(ii) Hence find the volume of the ring **3**

(c) The region under the curve $y = e^{-x^2}$ and above the x-axis is rotated about the y axis for $-a \le x \le a$ to form a solid as shown below.



(i) Divide the resulting solid into cylindrical shells S of radius t as shown in the diagram and show each shell S has an approximate volume given by $\delta V = 2\pi t e^{-t^2}\delta t$, where δt is the thickness of the shell. **2**

(ii) Hence calculate the volume of the solid. **2**

(iii) What is the limiting value of the volume of the solid as $a \to \infty$? **2**

Question 7. (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 (1-x^2)^n \, dx$.

(i) Show by using integration by parts $I_n = \dfrac{2n}{2n+1} I_{n-1}$ for $n = 0, 1, 2, 3, \ldots$ **3**

(ii) Hence evaluate $\int_0^1 (1 - x^2)^4 dx$ **3**

(b) A special dish is designed by rotating the region bounded by the curve $y = 2\cos x$ $(0 \le x \le 2\pi)$ and the line $y = 2$ through 360° about the y axis.

i) Use the method of cylindrical shells to show that the volume of the dish is given by **3**

$$4\pi \int_0^{2\pi} x(1 - \cos x) dx .$$

ii) Hence find the volume. **3**

(c) The polynomial $P(x)$ is given by $P(x) = 2x^3 - 9x^2 + 12x - k$, where k is real. **3**

Find the range of values for k for which $P(x) = 0$ has 3 real roots.

Question 8. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use integration by parts to find $\int \sin^{-1} x \ \ dx$. **3**

(b) (i) Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **3**

(iv) Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots **3**

$$x_1 = \cos\frac{\pi}{12},\ x_2 = -\cos\frac{\pi}{12},\ x_3 = \cos\frac{5\pi}{12},\ x_4 = -\cos\frac{5\pi}{12}$$

(iii) Hence show that $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$ **2**

(c) $P(x)$ is a polynomial of degree n with rational coefficients. **4**

If the leading coefficient is a_0 and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of $P(x) = 0$ prove that:

$$P'(x) = \frac{P(x)}{x-\alpha_1} + \frac{P(x)}{x-\alpha_2} + \frac{P(x)}{x-\alpha_3} + ...\frac{P(x)}{x-\alpha_n}$$