2008

TRIAL

HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

Total marks-120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Question 1 (15 Marks)

a) $\int \frac{x\,dx}{\sqrt{9-4x^2}}$ **2**

b) $\int \frac{dx}{\sqrt{9-4x^2}}$ **2**

c) Use integration by parts to evaluate $\int_1^e x^3 \ln x\,dx$ **3**

d) (i) Find real numbers a,b and c such that $\frac{5x^2-4x-9}{(x-2)(x^2-3)} = \frac{a}{x-2} + \frac{bx+c}{x^2-3}$ **2**

(ii) Hence show that $\int_3^4 \frac{5x^2-4x-9}{(x-2)(x^2-3)} = \ln\frac{52}{3}$ **2**

e) $\int \sec^3 x \tan x\,dx$ **2**

f) $\int \frac{dx}{x^2+4x+13}$ **2**

Question 2 (15 marks)

a) Let $z = 2 + i$ and $w = 3 - 4i$, find

(i) z^2 — 1

(ii) $\frac{1}{z}$ — 1

(iii) $w\bar{z}$ — 1

b) (i) Express $1-\sqrt{3}i$ in mod arg form — 2

(ii) Hence find $(1-\sqrt{3}i)^5$ — 1

(iii) Express $(1-\sqrt{3}i)^5$ in the form x + yi where x and y are real — 2

c) If u and v are two non zero complex numbers. Show that if $\frac{u}{v} = ik$ for some $k \in R$

(i) $\bar{u}v + \bar{v}u = 0$ — 2

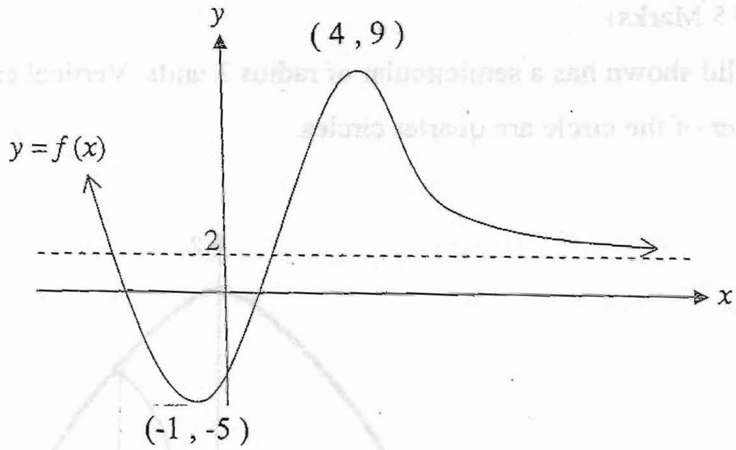
(ii) If $\bar{u}v + \bar{v}u = 0$ what is the relationship between arg v and arg u — 2

d) If ω is a complex root of the equation $z^3 = 1$

(i) Show that $1+\omega+\omega^2 = 0$ — 1

(ii) Find the value of $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ — 2

Question 3 (15 Marks)



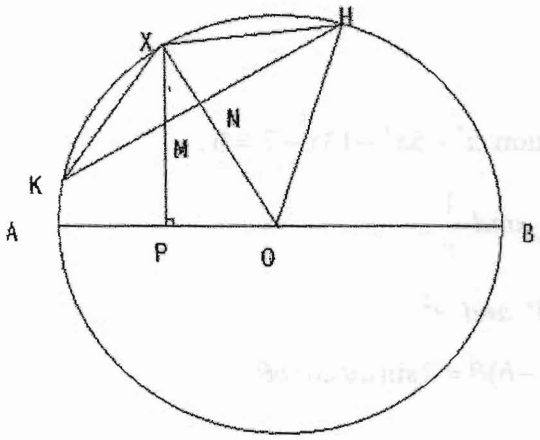
a) The graph of $y = f(x)$ is shown above. It has been reproduced for you on pages 9 and 10, detach these pages and draw neat sketches of the following. Include these pages in your solutions. The point of intersection of *f(x)* and the asymptote is (1 , 2).

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = f(|x|)$ 2

(iii) $y = f'(x)$ 2

(iv) $y = f(\frac{1}{x})$ 2



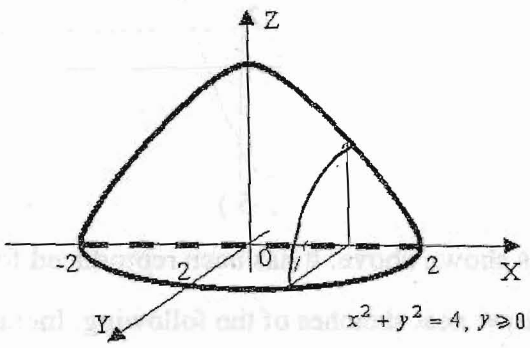
b) The circle above has diameter AB and centre O. KH is a chord to the circle and X is a point on the circumference such that KX = XH. XP is the perpendicular from P to AB. Prove that PNMO is a cyclic quadrilateral. 3

c) (i) Find the square root of $-8-8\sqrt{3}i$ 2

(ii) Hence solve the quadratic equation $x^2 - 2\sqrt{2}ix + 2\sqrt{3}i = 0$ 2

Question 4 (15 Marks)

a) The solid shown has a semicircular of radius 2 units. Vertical cross sections perpendicular to the diameter of the circle are quarter circles.



(i) By slicing at right angles to the x-axis show that the volume is given by

$$V=\frac{\pi}{2}\int_0^2 4-x^2dx$$ 2

(ii) Find the volume 2

b) The region bounded by the curve $y=\sin^{-1}x$ and the x-axis in the first quadrant is rotated about the line y = - 1. Using the method of cylindrical shells find the volume of the shape formed. 4

c) Let α,β *and* γ be the roots of the cubic equation $x^3-5x^2+13x-7=0$.

(i) Find the polynomial with roots $\frac{1}{\alpha},\frac{1}{\beta}$ *and* $\frac{1}{\gamma}$ 2

(ii) Find the polynomial with roots α^2,β^2 *and* γ^2 2

d) (i) Prove the identity $\sin(a+b)\theta+\sin(a-b)\theta=2\sin a\theta\cos b\theta$ 1

(ii) Hence find $\int\sin 4\theta\cos 2\theta\, d\theta$ 2

Question 5 (15 Marks)

a) Given the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Find:

(i) the eccentricity. 2

(ii) The coordinates of the foci 1

(iii) The equation of the directrices 1

(iv) Sketch the ellipse showing all essential features. 2

b) Given the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$

(i) Show that the point P with coordinates $(3\sec\theta, 2\tan\theta)$ lies on the hyperbola 1

(ii) Find the equation of the normal to the hyperbola at P. 2

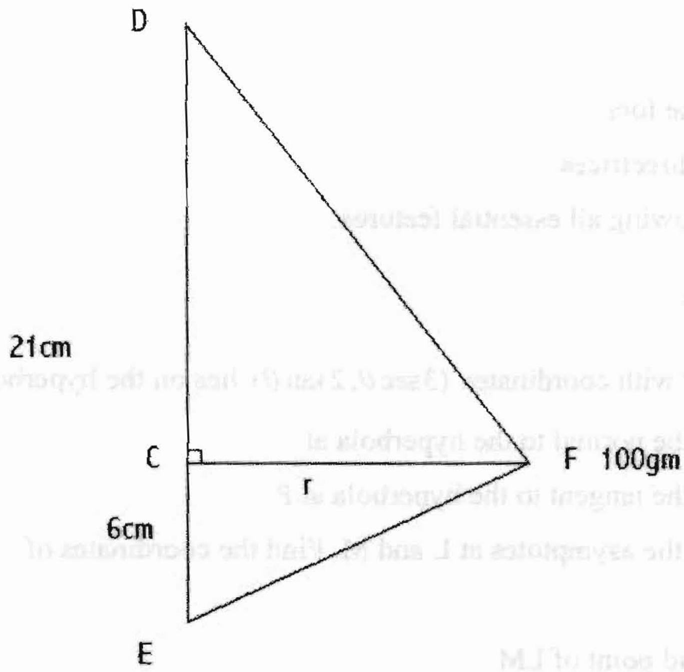
(iii) Find the equation of the tangent to the hyperbola at P. 2

(iv) The tangent at P cuts the asymptotes at L and M. Find the coordinates of L and M. 2

(v) Show that P is the mid point of LM. 2

Question 6. (15 Marks)

a)



A light inelastic string of length 27cm is attached to two points D and E on the vertical shaft DE, distance 21cm apart, E being vertically below D. F is a smooth ring of mass 100gms threaded on the string. The system is such that F moves with constant speed in a horizontal circle 6cm above E.

(i) Find the lengths of DF, FE and r. 3

(ii) Find the tension in the string. 2

(iii) Find the angular speed of F about DE 2

b) A bullet is fired vertically into the air with a speed of 800m/s. In the air the bullet experiences air resistance equal to $\frac{mv}{5}$ as well as gravity.

(i) Find the height reached to the nearest metre. 2

(ii) The time taken to achieve this height. 2

(iii) As the bullet returns to the ground it is subject to the same forces, Find the terminal velocity. 2

c) Solve for x $\tan^{-1} 3x - \tan^{-1} 2x = \tan^{-1}\frac{1}{5}$ 2

Question 7 (15 Marks)

a) The cubic equation $x^3 - 3x - 1 = 0$ is solved in two steps. Firstly let $x = u + v$ and secondly solve the quadratic equation $\lambda^2 - \lambda + 1 = 0$ the roots of which are u^3 and v^3.

(i) Solve the quadratic equation for u^3 and v^3. 1

(ii) Use De Moivre's theorem to find the cube roots with the arguments of least magnitude. 3

(iii) Find the value of x leave in trigonometric form. 1

b) Let $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$ n = 0,1,2,3 ...

(i) Show that $I_n = \dfrac{2n}{2n+3} I_{n-1}$ 2

(ii) Hence evaluate $\int_0^1 x^3 \sqrt{1-x}\,dx$ 2

(iii) Show that $I_n = \dfrac{n!(n+1)!}{(2n+3)!} 4^{n+1}$ 2

c) The curves y = cos x and y = tan x intersect at a point P whose x coordinate is α

(i) Show that the curves intersect at right angles at P. 2

(ii) Show that $\sec^2 \alpha = \dfrac{1+\sqrt{5}}{2}$ 2

Question 8 (15 Marks)

a) If $U_1 = \sqrt{2}$ and $U_n = \sqrt{2+U_{n-1}}$ Prove by Mathematical Induction that

$U_n < \sqrt{2} + 1$ for all n. 3

b) (i) Sketch the graph of $y = \frac{1}{x}$. With the aid of your sketch, show that for any

positive number u, $\frac{u}{1+u} < \int_1^{1+u} \frac{1}{x}\,dx < u$ 1

(ii) Deduce from (i) that $\frac{1}{1+r} < \ln\frac{r+1}{r} < \frac{1}{r}$, where $r > 0$ 1

(iii) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n} - \ln n$. By using (ii) show that

$\frac{1}{n} < a_n < 1$ 2

c) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 2

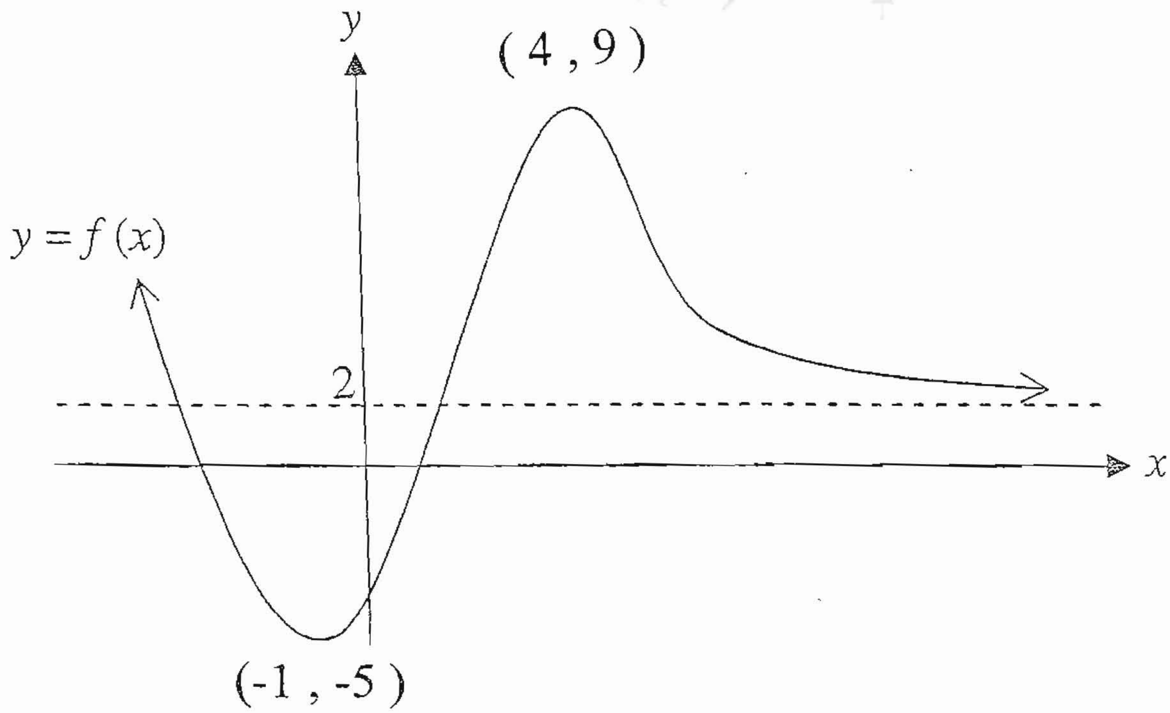
(ii) Hence find the value of $\int_0^\pi x \sin x \, dx$ 2

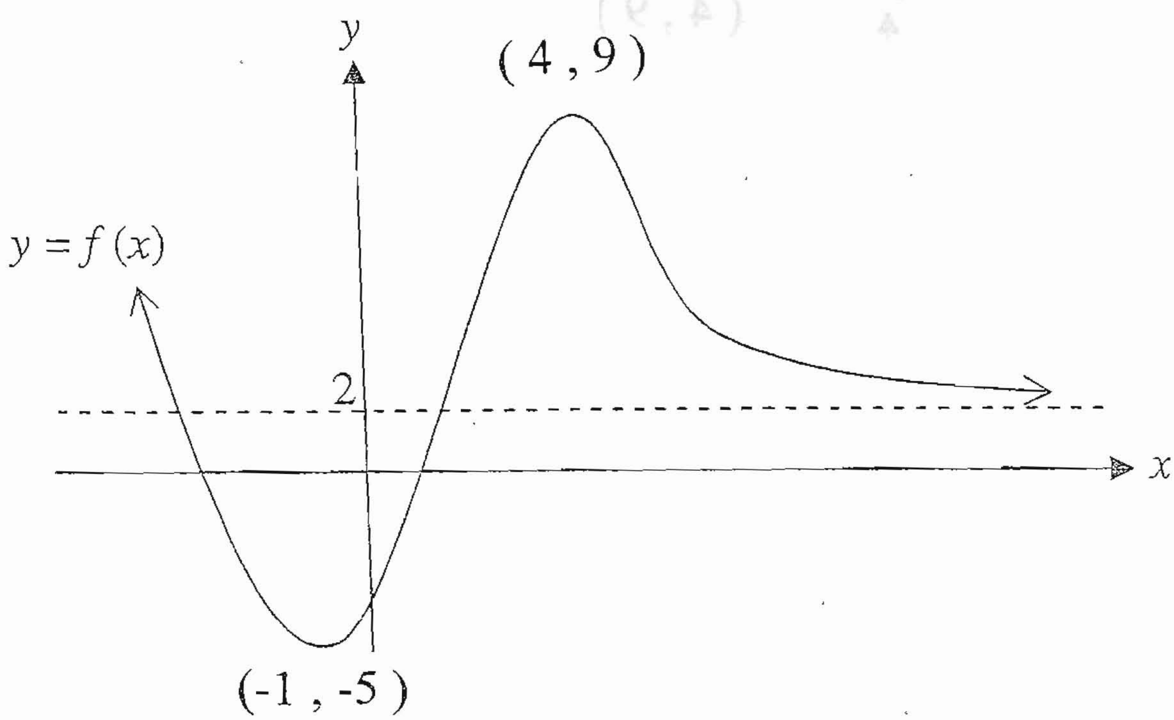
d) (i) Show that the gradient function for $x^2 + y^2 + xy = 12$ is

$$\frac{dy}{dx} = \frac{-(2x+y)}{2y+x}$$ 2

(ii) Find the coordinates of the stationary points of this function 1

(iii) Find the coordinates of the points of contact of any vertical tangents. 1


y
(4 , 9)
y = f (x)
2
x
(-1 , -5)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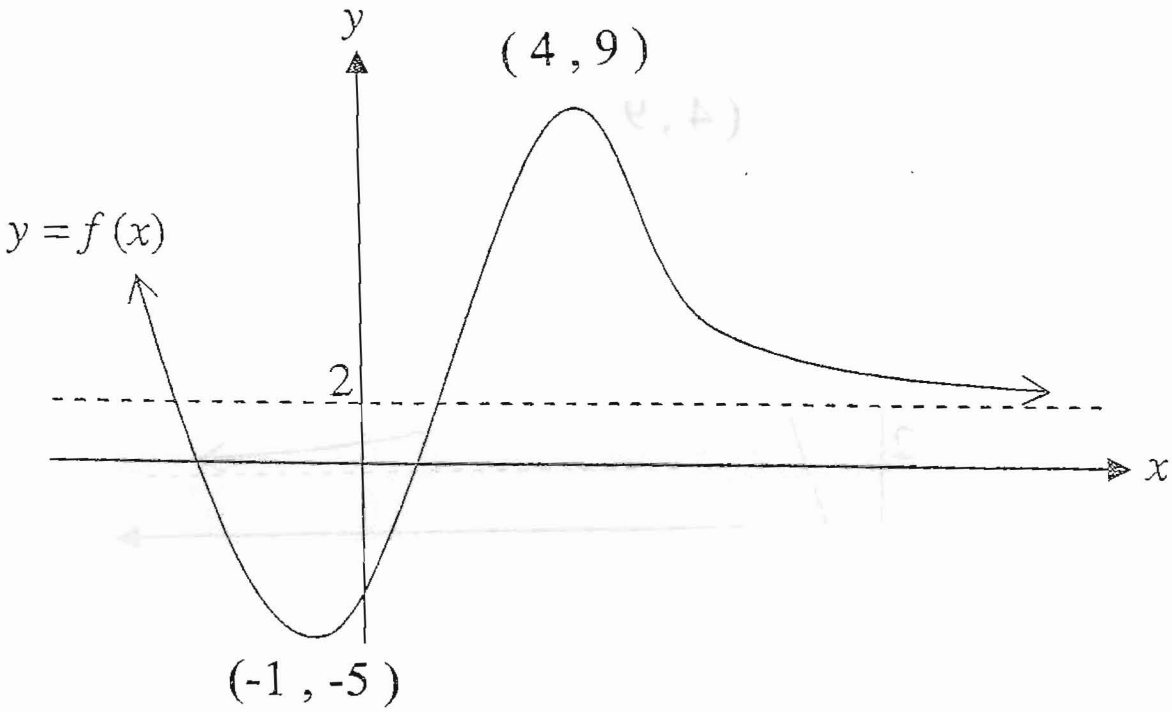
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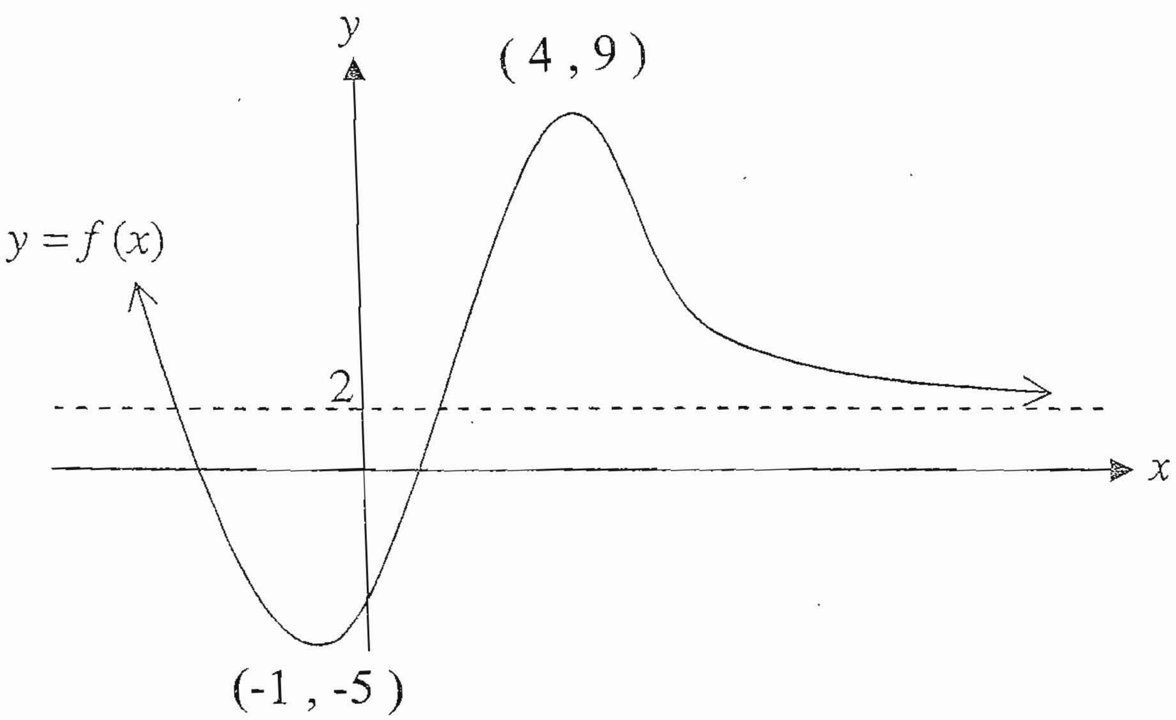
y
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y = f (x)
2
x
(-1 , -5)

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2008 TRIAL EXT 2

Q1 (a) $I = \int \frac{x\,dx}{\sqrt{9-4x^2}}$

let $u = 9-4x^2$

$\frac{du}{dx} = -8x$

$du = -8x\,dx$

$I = -\frac{1}{8}\int \frac{-8x\,dx}{\sqrt{9-4x^2}}$

$= -\frac{1}{8}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{8}\left(2u^{1/2}\right) + C$

$= -\frac{1}{4}\sqrt{9-4x^2} + C$ (2)

(b) $I = \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9/4-x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + C.$ (2)

(c) $I = \int_1^e x^3 \ln x\,dx$

$\int_1^e dv\,u = [uv]_1^e - \int_1^e v\,du$

let $u = \ln x$ $\quad dv = x^3$

$du = 1/x$ $\quad v = x^4/4$

$= \left[(\ln x)\frac{x^4}{4}\right]_1^e - \int_1^e \frac{x^4}{4}\frac{1}{x}dx$

$= \frac{e^4}{4} - \int_1^e \frac{x^3}{4}dx$

$= \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e$

$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}$

$= \frac{3e^4}{16} + \frac{1}{16} = \frac{1}{16}\left(3e^4+1\right)$ (3)

(d) (i) $\frac{5x^2-4x-9}{(x-2)(x^2-3)} = \frac{a}{x-2} + \frac{bx+c}{x^2-3}$

$\therefore 5x^2-4x-9 = a(x^2-3) + (bx+c)(x-2)$

let $x = 2$ $\quad a = 3$

$x = 0$ $\quad c = 0$

by coefficient of x^2 $\quad b = 2.$ (2)

(ii) $\int_3^4 \frac{5x^2-4x-9}{(x-2)(x^2-3)} = \int_3^4 \frac{3}{x-2} + \frac{2x}{x^2-3}$

$= \left[3\ln(x-2) + \ln(x^2-3)\right]_3^4$

$= \left[3\ln 2 + \ln(13) - 3\ln 1 - \ln 6\right]$

$= \ln\frac{8\times 13}{6} = \ln\frac{4\times 13}{3}$

$= \ln\frac{52}{3}.$ (2)

(e) $\int \sec^3 x \tan x\,dx$

let $u = \sec x$

$\frac{du}{dx} = \sec x \tan x\,dx$

$\int u^2\,du = u^3/3 + C$

$= \frac{\sec^3 x}{3} + C$ (2)

(f) $\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{x^2+4x+4+9}$

$= \int \frac{dx}{(x+2)^2+3^2}$

$= \frac{1}{3}\tan^{-1}\frac{x+2}{3} + C.$ (2)

Q2.

(a) (i) $Z^2 = (2+i)^2$

$= 3+4i$. (1)

(ii) $\frac{1}{Z} = \frac{\bar{Z}}{Z\bar{Z}} = \frac{2-i}{(2+i)(2-i)}$

$= \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5}.$ (1)

(iii) $\omega\bar{Z} = (3-4i)(2-i)$

$= 2-11i$ (1)

(b) (i) $1-\sqrt{3}i$ $\quad$ mod $\sqrt{1^2+(-\sqrt{3})^2}$

mod $= 2.$ (2)

arg $= \tan^{-1}\frac{-\sqrt{3}}{1} = -\pi/3.$

(ii) $(1-\sqrt{3}i)^5 = 2^5 \operatorname{cis} 5(-\pi/3)$

$= 32 \operatorname{cis} -5\pi/3$ (1)

better $= 32 \operatorname{cis} \pi/3$

(iii) $(1-\sqrt{3}i)^5 = 32(\cos\pi/3 + i\sin\pi/3)$

$= 16 + 16\sqrt{3}i$ (2)

c) (i) $\bar{u}v + \bar{v}u = 0$

$\frac{u}{v} = ik.$

$\therefore \overline{\left(\frac{u}{v}\right)} = -ik$

$\frac{\bar{u}}{\bar{v}} = -ik$

$\frac{\bar{v}}{\bar{u}} = \frac{-1}{ik}$

$\frac{u}{v}\times\frac{\bar{v}}{\bar{u}} = -1$

$u\bar{v} = -v\bar{u}$

$u\bar{v} + v\bar{u} = 0.$ (2)

(ii) $\bar{u}v + \bar{v}u = 0$

$\therefore \frac{u}{v} = ik$

$\arg\frac{u}{v} = ik$

$\arg\frac{u}{v} = \pm\pi/2$

$\arg u - \arg v = \pm\pi/2$

$\therefore$ Difference of arg u and arg v is $\pi/2$. (2)

(d) (i) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

if ω is a root

$(\omega-1)(\omega^2+\omega+1) = 0$

but ω is complex $\therefore \omega - 1 \neq 0$

$\therefore \omega^2+\omega+1 = 0$ (1)

or by sum of roots

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)$

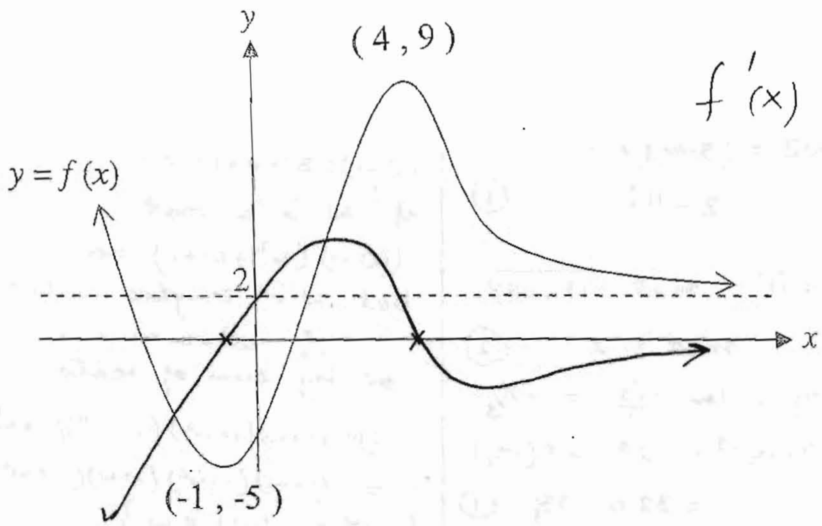
$\{\omega^4 = \omega(\omega^3) = \omega\}$

$= \{(1+\omega)(1+\omega^2)\}^2$

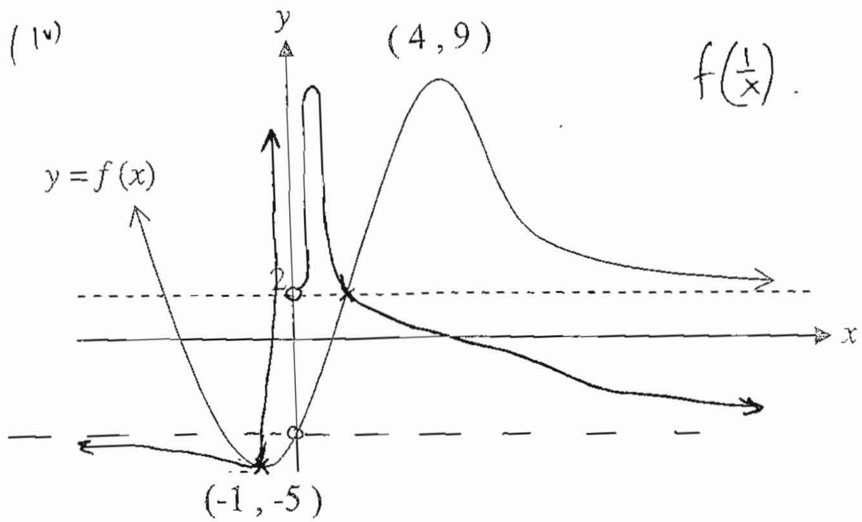
$= (1+\omega+\omega^2+\omega^3)^2$

$= 1^2 = 1$ (2)

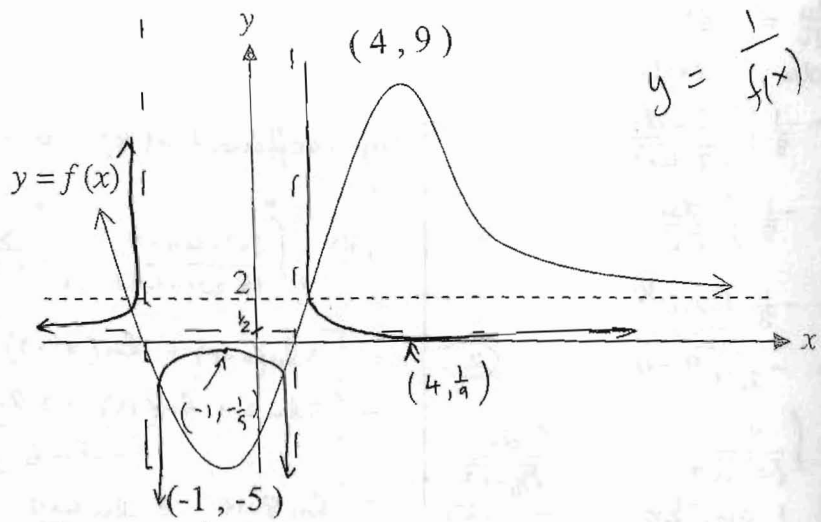
(iii)


y
(4 , 9)
f '(x)
y = f (x)
2
x
(-1 , -5)

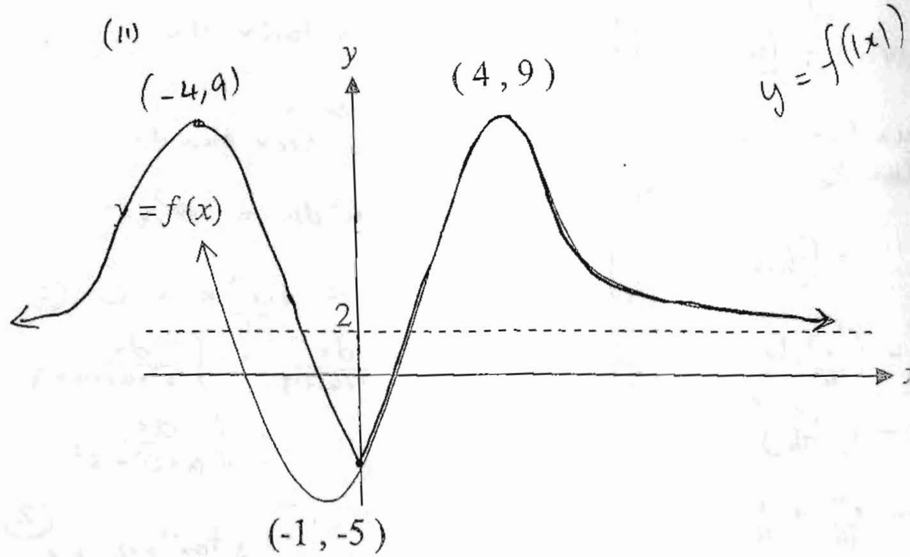
(iv)


y
(4 , 9)
f(1/x)
y = f (x)
2
x
(-1 , -5)

(a) (i)


y
(4 , 9)
y = 1/f(x)
y = f (x)
2
1/2
x
(-1, -1/5)
(4, 1/9)
(-1 , -5)

(ii)


y
(-4, 9)
(4 , 9)
y = f(|x|)
y = f (x)
2
x
(-1 , -5)

Question 3.

(b) $\angle XOH = 2\angle XKH$ (α) (angle at centre)

$\angle XHK = \angle XKH$ (β) ($\triangle XKH$ is isosceles)

$OX = OH$ (radii)

$\therefore \angle OXH = \angle OHX$

$\angle XOH + 2\angle OXK = 180$

$\therefore 2\angle XKH + 2\angle OXK = 180$ (α)

$\angle XKH + \angle OXK = 90°$

$\therefore \angle XHK + \angle OXK = 90°$ (β)

$\therefore \angle XNH = 90°$ (3rd $\angle$ of $\triangle XNH$)

$\therefore$ PNMO is cyclic (3)

$\angle NPO + \angle ONM = 180°$.

c (i) $\sqrt{-8-8\sqrt{3}i} = a+bi$

$-8-8\sqrt{3}i = a^2-b^2+2abi$

$-8 = a^2-b^2$

$-4\sqrt{3} = ab$. (2)

where $a = \pm 2 \quad b = \mp 2\sqrt{3}$

(ii) $x^2 - 2\sqrt{2}ix + 2\sqrt{3}i = 0$

$$x = \frac{2\sqrt{2}i \pm \sqrt{-8 - 8\sqrt{3}i}}{2}$$

$$x = \frac{2\sqrt{2}i \pm (2-2\sqrt{3}i)}{2}$$

$$x = \frac{2+2\sqrt{2}i-2\sqrt{3}i}{2}, \frac{-2+2\sqrt{2}i+2\sqrt{3}i}{2}$$

$$x = 1+(\sqrt{2}-\sqrt{3})i, \quad -1+(\sqrt{2}+\sqrt{3})i$$ (2)

Question 4

(a)(i) $A_{\frac{1}{4}\text{ circle}} = \frac{1}{4}\pi r^2$

$= \frac{1}{4}\pi y^2$

$\delta V = \frac{1}{4}\pi y^2 \delta x$

$$V = \lim_{\delta x \to 0} 2\sum_{x=0}^{2} \tfrac{1}{4}\pi y^2 \delta x$$

$$V = \frac{1}{2}\pi \int_0^2 y^2\,dx$$

but $x^2+y^2=4$

$y^2 = 4-x^2$

$$V = \frac{\pi}{2}\int_0^2 4-x^2\,dx$$ (2)

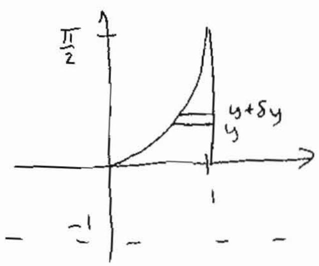
(ii) $$V = \frac{\pi}{2}\int_0^2 4-x^2\,dx$$

$$= \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{\pi}{2}\left[(8-\tfrac{8}{3}) - (0)\right]$$

$$= \frac{8\pi}{3}\ u^3$$ (2)

(b)



δV = Vol outer cylinder − Volume inner cylinder

$= \pi R^2 h - \pi r^2 h$

$= \pi\left((y+1+\delta y)^2 - (y+1)^2\right) 1-x$

$= \pi(1-x)\left\{(y+1)^2 + 2(y+1)\delta y + \delta y^2 - (y+1)^2\right\}$

$\delta V = 2\pi(1-x)(y+1)\delta y$

$$V = \lim_{\delta y \to 0} 2\pi \sum_{y=0}^{\pi/2} (1-x)(y+1)\delta y$$

$$V = 2\pi \int_0^{\pi/2} (1-x)(y+1)\,dy$$

but $y = \sin^{-1}x$

$\therefore x = \sin y$

$$V = 2\pi \int_0^{\pi/2} (1-\sin y)(y+1)\,dy$$

$$= 2\pi \int_0^{\pi/2} y+1-\sin y\,dy - 2\pi \int_0^{\pi/2} y\sin y\,dy$$

$$V_A = 2\pi\left[\tfrac{y^2}{2} + y + \cos y\right]_0^{\pi/2}$$

$$= 2\pi\left[\left(\tfrac{\pi^2}{8} + \tfrac{\pi}{2} + 0\right) - (0+1)\right]$$

$$= 2\pi\left[\tfrac{\pi^2}{8} + \tfrac{\pi}{2} - 1\right]$$

$$V_B = -2\pi\int_0^{\pi/2} y\sin y\,dy$$

by parts

$$= -2\pi\left\{\left[-y\cos y\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos y\right\}$$

$$= -2\pi\left\{(0) + \left[\sin y\right]_0^{\pi/2}\right\}$$

$= -2\pi$.

$V_{TOT} = V_A + V_B$ (4)

$$= 2\pi\left[\tfrac{\pi^2}{8} + \tfrac{\pi}{2} - 2\right] u^3$$

(c) $f(x) = x^3 - 5x^2 + 13x - 7$

$$f\left(\tfrac{1}{x}\right) = \frac{1}{x^3} - \frac{5}{x^2} + \frac{13}{x} - 7$$

$\times x^3 \qquad = 1 - 5x + 13x^2 - 7x^3$

$f(x) = 7x^3 - 13x^2 + 5x - 1$ (2)

Question 4 (cont.)

(ii) roots, α^2, β^2 and γ^2

consider the function

$f(\sqrt{x}) = (\sqrt{x})^3 - 5x + 13\sqrt{x} - 7$

$0 = x\sqrt{x} - 5x + 13\sqrt{x} - 7$

$(5x+7)^2 = (x\sqrt{x} + 13\sqrt{x})^2$

$25x^2 + 70x + 49 = x^3 + 26x^2 + 169x$

$0 = x^3 + x^2 + 99x^2 - 49.$

$f(x) = x^3 + x^2 + 99x^2 - 49.$

(d) (i) $\sin(a+b)\theta$

$= \sin a\theta \cos b\theta + \cos a\theta \sin b\theta.$

$\sin(a-b)\theta$

$= \sin a\theta \cos b\theta - \cos a\theta \sin b\theta.$

$\therefore \sin(a+b)\theta + \sin(a-b)\theta$

$= 2\sin a\theta \cos b\theta$ (1)

(ii) $\int \sin 4\theta \cos 2\theta \, d\theta$

$a = 4 \quad b = 2$

$= \frac{1}{2}\int \sin 6\theta + \sin 2\theta \, d\theta$

$= \frac{1}{2}\left[-\frac{\cos 6\theta}{6} - \frac{\cos 2\theta}{2}\right] + C$

$= -\frac{1}{12}\left[\cos 6\theta + 3\cos 2\theta\right] + C$ (2)

Question 5.

(a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(i) $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$4/9 = 1 - e^2$

$e = \sqrt{5}/3.$ (2)

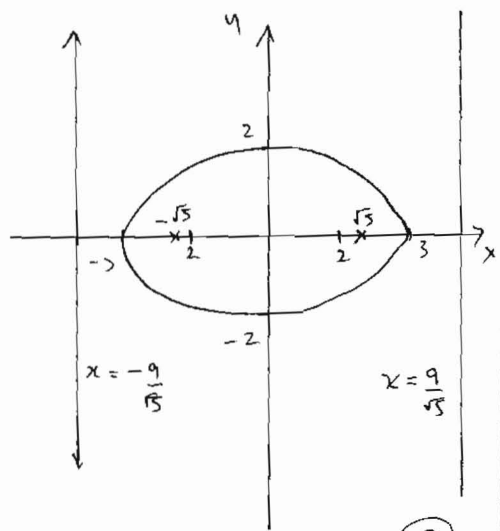
(ii) Foci $(\pm ae, 0)$

$(\pm\sqrt{5}, 0)$ (1)

(iii) DIRECTRIX $x = \pm 9/e$

$x = \pm \frac{3}{\sqrt{5}/3}$

$x = \pm 9/\sqrt{5}$ (1)



(2)

(b) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

(i) $\frac{(3\sec\theta)^2}{9} - \frac{(2\tan\theta)^2}{4} = 1$

$\sec^2\theta - \tan^2\theta = 1$

$\sec^2\theta = \tan^2\theta + 1$

true for all θ (pythag)

(ii) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Diff $\frac{2x}{9} - \frac{2y}{4}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{4x}{9y}$

$\frac{dy}{dx} = \frac{4(3\sec\theta)}{9(2\tan\theta)}$

$\therefore$ GRADIENT NORMAL $= -\frac{3\tan\theta}{2\sec\theta}$

EQ^N OF NORMAL

$y - 2\tan\theta = -\frac{3\tan\theta}{2\sec\theta}(x - 3\sec$

$\frac{y}{3\tan\theta} - \frac{2}{3} = -\frac{x}{2\sec\theta} + \frac{3}{2}$

$\frac{3x}{\sec\theta} + \frac{2y}{\tan\theta} = 13$ (2)

(iii) EQ^N OF TANGENT

$y - 2\tan\theta = \frac{2\sec\theta}{3\tan\theta}(x - 3\sec\theta)$

$3\tan\theta\, y - 6\tan^2\theta = 2\sec\theta x - 6\sec$

$6\sec^2\theta - 6\tan^2\theta = 2\sec\theta x - 3\tan$

$6 = 2\sec\theta x - 3\tan\theta$

$1 = \frac{\sec\theta\, x}{3} - \frac{\tan\theta\, y}{2}$

1 Question 5 continued

(iv) Asymptotes

$$y = \pm \frac{b}{a} x$$

$$y = \frac{2}{3} x \qquad y = -\frac{2}{3} x$$

$$1 = \frac{\sec\theta}{3} x - \frac{\tan\theta}{2} y \qquad (\alpha)$$

$$y = \tfrac{2}{3} x \qquad (\beta)$$

$$L \left(\frac{3}{\sec\theta - \tan\theta} , \frac{2}{\sec\theta - \tan\theta} \right)$$

$$M \left(\frac{3}{\sec\theta + \tan\theta} , \frac{-2}{\sec\theta + \tan\theta} \right)$$ (2)

(v) MID PT LM

$$x = \frac{1}{2} \left(\frac{3}{\sec\theta - \tan\theta} + \frac{3}{\sec\theta + \tan\theta} \right)$$

$$x = \frac{1}{2} \left(\frac{3\sec\theta + 3\tan\theta + 3\sec\theta - 3\tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)} \right)$$

$$x = \frac{3\sec\theta}{\sec^2\theta - \tan^2\theta}$$

$$x = 3\sec\theta$$

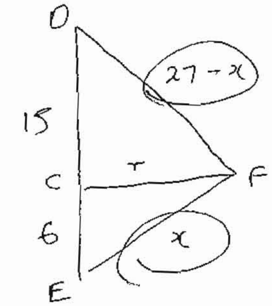
$$y = \frac{1}{2} \left(\frac{2}{\sec\theta - \tan\theta} + \frac{-2}{\sec\theta + \tan\theta} \right)$$

$$y = \frac{1}{2} \left(\frac{2\sec\theta + 2\tan\theta - 2\sec\theta + 2\tan\theta}{\sec^2\theta - \tan^2\theta} \right)$$

$$y = 2\tan\theta$$

$\therefore$ P is mid pt of LM. (2)

Question 6.



$$DF^2 = r^2 + 15^2$$

$$FE^2 = r^2 + 6^2$$

$$(27 - x)^2 = r^2 + 225. \qquad (\alpha)$$

$$x^2 = r^2 + 36 \qquad (\beta)$$

$$729 - 54x + x^2 = r^2 + 225 \qquad (\alpha)$$

subst (β)

$$729 - 54x + r^2 + 36 = r^2 + 225.$$

$$54x = 540$$

$$x = 10$$

$$\therefore EF = 10 \text{ cm}$$

$$DF = 17 \text{ cm}$$

$$r = 8 \text{ cm}$$ (3)

Resolving vertically at F

$$Mg = T\cos\angle D - T\cos\angle E$$

$$0.1 \times 9.8 = T\tfrac{15}{17} - T\tfrac{3}{5}.$$

$$T = 3.54 \text{ N}.$$ (2)

Resolving horizontally at F

$$m\omega^2 r = T\sin\angle D + T\sin\angle E$$

$$0.1 \times \omega^2 \times 0.08 = 3.54 (4/5 + 8/17)$$

$$\omega^2 = 563.5 \text{ rad/sec}$$

$$\omega = 23.7 \text{ rad/sec}$$

(b) (i) $F = ma$

$$ma = -\frac{mv}{5} - mg$$

$$a = -\left(\frac{v + 5g}{5} \right)$$

FOR HEIGHT USE $a = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -\frac{(v+5g)}{5}$$

$$\frac{dv}{dx} = -\frac{v + 5g}{5v}$$

$$\frac{dx}{dv} = -5 \left(\frac{v}{v+5g} \right)$$

$$\int_0^H dx = -5 \int_{800}^{0} \frac{v}{v+5g} \, dv$$

$$H = 5 \int_0^{800} 1 - \frac{5g}{v+5g} \, dv$$

$$H = 5 \Big[v - 5g \ln(v + 5g) \Big]_0^{800}$$

using $g = 10$.

$$H = 5 \Big[(800 - 50 \ln 850 + 50 \ln 50) \Big]$$

$$H = 4000 + 250 \ln \tfrac{50}{850}$$

$$H \doteqdot 3292 \text{ m}$$ (2)

(ii) FOR TIME USE $\frac{dv}{dt} = a$

$$\frac{dv}{dt} = -\frac{(v+5g)}{5}$$

$$\frac{dt}{dv} = \frac{-5}{v+5g}$$

$$dt = -\frac{5\,dv}{v+5g}$$

Question 6 cont.

$$\int_0^t dt = -5\int_{800}^{0} \frac{dv}{v+5g}$$

$$t = 5\left[\ln(v+5g)\right]_0^{800}$$

$$t = 5(\ln 850 - \ln 50)$$ (2)

$$t = 5\ln 17 \doteqdot 14.17 \text{ secs}$$

(iii) $F = ma$

$$ma = mg - \frac{mv}{5}$$

$$a = \frac{5g - v}{5}$$

FOR TERMINAL VELOCITY

$$a = 0$$

$$5g = v$$

$$v = 50\text{ m/sec}$$ (2)

(c) $\tan^{-1}3x - \tan^{-1}2x = \tan^{-1}\frac{1}{5}$

let $\alpha = \tan^{-1}3x \quad 3x = \tan\alpha$

let $\beta = \tan^{-1}2x \quad 2x = \tan\beta$.

$$\tan(\alpha - \beta) = \tan(\tan^{-1}3x - \tan^{-1}2x)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan\left(\tan^{-1}\frac{1}{5}\right) = \frac{3x - 2x}{1 + (3x)(2x)}$$

$$\frac{1}{5} = \frac{x}{1 + 6x^2}$$

$$+6x^2 + 1 = 5x$$

$$0 = 6x^2 - 5x + 1$$

$$0 = (3x - 1)(2x - 1)$$

$$x = \frac{1}{2}, \frac{1}{3}.$$ (2)

Question 7

(a) (i) $\lambda^2 - \lambda + 1 = 0$

$$\lambda = \frac{+1 \pm \sqrt{1-4}}{2}$$

$$\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i$$ (1)

(ii) $\lambda = 1\operatorname{cis}\frac{\pi}{3}, \quad 1\operatorname{cis}-\frac{\pi}{3}$

$\therefore u^3 = 1\operatorname{cis}\frac{\pi}{3}, \quad v^3 = 1\operatorname{cis}-\frac{\pi}{3}$.

$$u = \operatorname{cis}\frac{2\pi k + \pi/3}{3} \quad v = \operatorname{cis}\frac{2\pi k - \pi/3}{3}$$

least arg

$$u = \operatorname{cis}\frac{\pi}{9} \quad v = \operatorname{cis}-\frac{\pi}{9}.$$ (3)

(iii) $x = u + v$

$$x = \cos\frac{\pi}{9} + i\sin\frac{\pi}{9} + \cos-\frac{\pi}{9} + i\sin-\frac{\pi}{9}$$

$$x = 2\cos\frac{\pi}{9}.$$ (1)

(b) $I_n = \int_0^1 x^n\sqrt{1-x}\,dx$

let $u = x^n$ let $dv = (1-x)^{1/2}$

$$\frac{du}{dx} = nx^{n-1} \quad v = -\tfrac{2}{3}(1-x)^{3/2}$$

$$\int_0^1 x^n\sqrt{1-x}\,dx = \left[uv\right]_0^1 - \int_0^1 v\frac{du}{dx}$$

$$= \left[x^n(1-x)^{3/2}\left(-\frac{2}{3}\right)\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)^{3/2}(x^{n-1})\,n$$

$$= 0 + \frac{2n}{3}\int_0^1 \sqrt{1-x}\,(1-x)\,x^{n-1}$$

$$= \frac{2n}{3}I_{n-1} - \frac{2n}{3}\int_0^1 \sqrt{1-x}\,x^n$$

$$\therefore \frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$$

$$\therefore I_n = \frac{2n}{2n+3}I_{n-1}.$$ (2)

(ii) $\int_0^1 x^3\sqrt{1-x}\,dx = \frac{6}{9}I_2$

$$I_2 = \frac{4}{7}I_1$$

$$I_1 = \frac{2}{5}I_0$$

$$I_0 = \int_0^1 \sqrt{1-x}\,dx$$

$$= \left[-\tfrac{2}{3}(1-x)^{3/2}\right]_0^1$$

$$= \tfrac{2}{3}.$$

$$I_3 = \frac{6}{9} \times \frac{4}{7} \times \frac{2}{5} \times \frac{2}{3}$$

$$= \frac{32}{315} = \frac{4^4\, 3!\, 4!}{9!}$$ (2)

(iii) $I_n = \frac{2n}{2n+3} \times \frac{2n-2}{2n+1} \cdots \frac{6}{9} \times \frac{4}{7} \times \frac{2}{5} \times$

$$= \frac{2^{n+1}\, n!}{(2n+3)(2n+1)\cdots 9\times 7\times 5\times 3\times 1}$$

$$= \frac{2^{n+1}\, n! \times (2n+2)(2n)\cdots 10\times 8\times 6}{(2n+3)!}$$

$$= \frac{2^{n+1}\, n!\,(n+1!)\,2^{n+1}}{(2n+3)!}$$

$$= \frac{n!\,(n+1!)\,4^{n+1}}{(2n+3)!}$$ (2)

c) (ii) $\cos x = \tan x$

$$\cos x = \frac{\sin x}{\cos x}$$

$$\cos^2 x = \sin x$$

$$1 - \sin^2 x = \sin x$$

$$0 = \sin^2 x + \sin x - 1$$

Question 7 cont.

$\sin x = \frac{-1 \pm \sqrt{1+4}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$

$\sin x = \frac{-1+\sqrt{5}}{2}$ acute x

but $\sin x = \cos^2 x$

$\cos^2 x = \frac{-1+\sqrt{5}}{2}$

$\therefore \sec^2 x = \frac{2}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}$

$= \frac{2(\sqrt{5}+1)}{4}$

$= \frac{1+\sqrt{5}}{2}$. (2)

(i) $y' = -\sin x$ $y' = \sec^2 x$

when $x = \alpha$

$(-\sin\alpha)(\sec^2\alpha) = \frac{1-\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2}$

$= \frac{1-5}{4} = -1$.

$\therefore$ product of gradients $= -1$

$\therefore$ $\cos x$ and $\tan x$

perpendicular at α (2)

Question 8.

a) $u_1 = \sqrt{2}$ $u_2 = \sqrt{2+u_{n-1}}$

Step 1 Prove true for $n=1$

LHS $= \sqrt{2}$

RHS $= \sqrt{2}+1$

LHS < RHS TRUE FOR $n=1$

Step 2. Assume true for $n=k$

$\therefore u_k = \sqrt{2+u_{k+1}}$

$u_k < \sqrt{2}+1$

Prove true for u_{k+1}

$u_{k+1} = \sqrt{2+u_k}$

$< \sqrt{2+\sqrt{2}+1}$

$< \sqrt{3+2\sqrt{2}}$

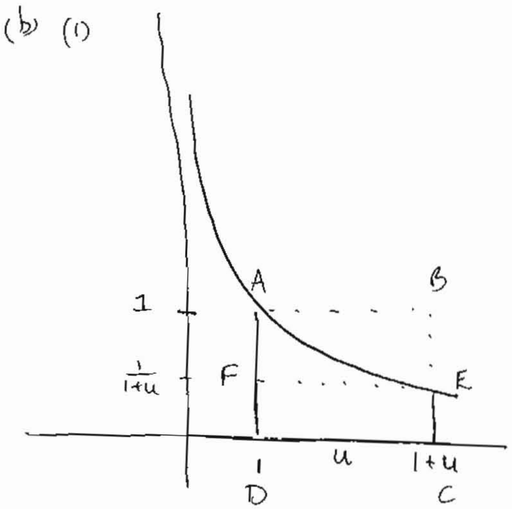
$= \sqrt{(\sqrt{2}+1)^2}$

$= \sqrt{2}+1$

$\therefore u_{k+1} < \sqrt{2}+1$

Step 3 By the principle of Mathematical Induction true for all n (3)

(b) (i)



By inspection

$A_{CDFE} < \int_1^{1+u} \frac{1}{x} < A_{ABCD}$

$u \times \frac{1}{1+u} < \int_1^{1+u} \frac{1}{x} < 1 \times u$

$\frac{u}{1+u} < \int_1^{1+u} \frac{1}{x}\,dx < u$. (1)

(ii) Now since

$\int_1^{1+u} \frac{1}{x}\,dx = [\ln x]_1^{1+u}$

$= \ln 1+u - \ln 1$

$= \ln 1+u$

$\frac{u}{1+u} < \ln 1+u < u$

let $u = 1/r$

$\frac{1/r}{1+\frac{1}{r}} < \ln 1+\frac{1}{r} < \frac{1}{r}$

$\frac{1}{r+1} < \ln \frac{r+1}{r} < \frac{1}{r}$ (1)

(iii) let $r = 1, 2, 3 \ldots n-1$

$\frac{1}{2} < \ln 2 - \ln 1 < 1$

$\frac{1}{3} < \ln 3 - \ln 2 < \frac{1}{2}$

$\vdots$

$\frac{1}{n} < \ln n - \ln(n-1) < \frac{1}{n-1}$

adding all three sides

$\frac{1}{2} + \frac{1}{3} \ldots \frac{1}{n} < \ln n - \ln 1 < 1 + \frac{1}{2} + \frac{1}{3} \ldots \frac{1}{n-1}$

so that

$\frac{1}{n} < 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n} - \ln n < 1$

$\frac{1}{n} < a_0 < 1$. (2)

(c) (i) $\int_0^a f(x)\,dx$

let $x = a - u$ when $x = a$, $u = 0$

$\frac{dx}{du} = -1$ when $x = 0$, $u = a$

Question 8 cont.

$$\int_0^a f(x)\,dx = \int_a^0 f(a-u)\,-du$$

$$= -\int_a^0 f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du$$

By change of variable

$$= \int_0^a f(a-x)\,dx$$

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \quad (2)$$

(ii) $\therefore \int_0^{\pi} x \sin x \, dx$

$$= \int_0^{\pi} (\pi - x) \sin(\pi - x)\,dx$$

N.B. $\sin(\pi - x) = \sin x$.

$$= \int_0^{\pi} \pi \sin x - x \sin x \, dx$$

$$\therefore 2\int_0^{\pi} x \sin x\,dx = \pi \int_0^{\pi} \sin x$$

$$\int_0^{\pi} x \sin x\,dx = \pi/2 \Big[-\cos x\Big]_0^{\pi}$$

$$= \pi/2\,[1+1]$$

$$= \pi \ u^2 \quad (2)$$

d) $x^2 + y^2 + xy = 12$

differentiate implicitly

$$2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$(2y + x)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{(2x+y)}{2y+x} \quad (2)$$

(ii) FOR STATIONARY POINTS

$$\frac{dy}{dx} = 0$$

$$2x + y = 0$$

$$y = -2x$$

Subst into function

$$x^2 + (-2x)^2 + x(-2x) = 12$$

$$x^2 + 4x^2 - 2x^2 = 12$$

$$3x^2 = 12$$

$$x^2 = 4$$

$x = 2 \quad y = -4 \qquad (2, -4)$

$x = -2 \quad y = 4 \qquad (2, +4)$ (1)

(iii) for vertical tangent

$$2y + x = 0$$

$$y = -x/2$$

$$x^2 + \left(\frac{-x}{2}\right)^2 + x\left(\frac{-x}{2}\right) = 12$$

$$x^2 + \frac{x^2}{4} - \frac{x^2}{2} = 12$$

$$\frac{3x^2}{4} = 12$$

$$x = \pm 4$$

$(4, -2) \qquad (-4, 2)$ (1)