Student Name: ______________________

Teacher: ______________________

2012
TRIAL HSC
EXAMINATION

Mathematics Extension 2

Examiners

Mr J. Dillon and Mr S. Gee

General Instructions

- Reading time - 5 minutes.
- Working time - 3 hours.
- Write using black or blue pen. Diagrams may be drawn in pencil.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11-16.
- Start each question in a separate answer booklet.
- Put your student number on each booklet.

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16. Each of these six questions are worth 15 marks
- Allow about 2 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Consider the hyperbola with the equation $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

What are the equations of the directrices?

(A) $y = \pm\frac{25}{13}$ (B) $y = \pm\frac{144}{13}$

(C) $x = \pm\frac{25}{13}$ (D) $x = \pm\frac{144}{13}$

2 The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi = -\frac{b^2}{a^2}$ (B) $\tan\theta\tan\phi = -\frac{a^2}{b^2}$

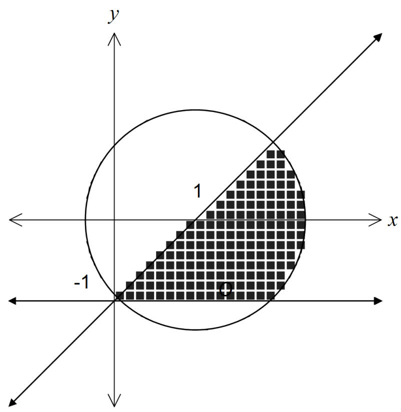
(C) $\tan\theta\tan\phi = \frac{b^2}{a^2}$ (D) $\tan\theta\tan\phi = \frac{a^2}{b^2}$

3 What is $-\sqrt{3} + i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ (B) $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ (D) $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

4 Consider the Argand diagram below.



Which inequality could define the shaded area?

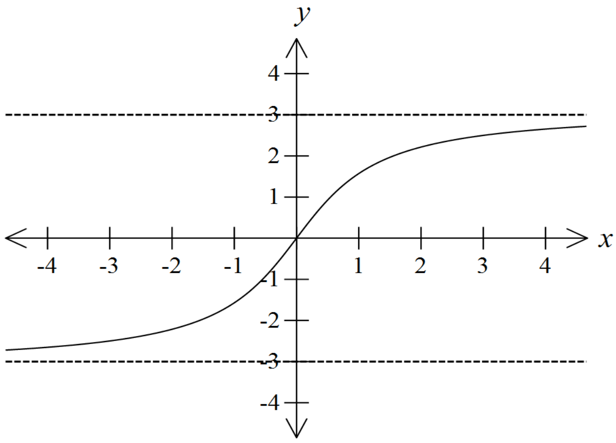
(A) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z-i) \leq \frac{\pi}{4}$

(B) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z+i) \leq \frac{\pi}{4}$

(C) $|z-1| \leq 1$ and $0 \leq \arg(z-i) \leq \frac{\pi}{4}$

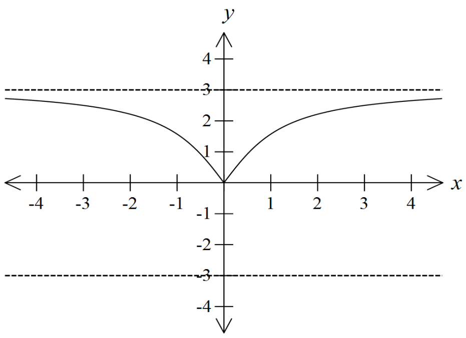
(D) $|z-1| \leq 1$ and $0 \leq \arg(z+i) \leq \frac{\pi}{4}$

5 The diagram shows the graph of the function $y = f(x)$.

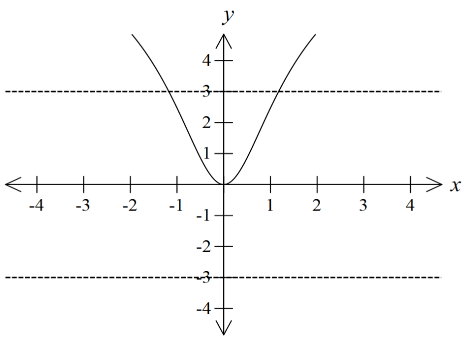


Which of the following is the graph of $y = \sqrt{f(x)}$?

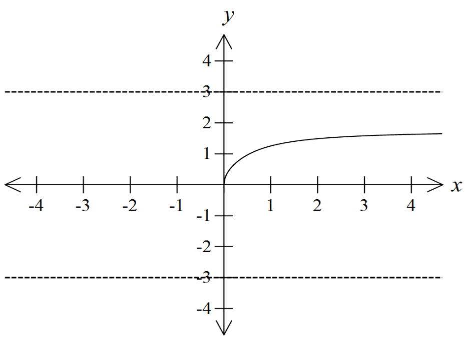
(A)



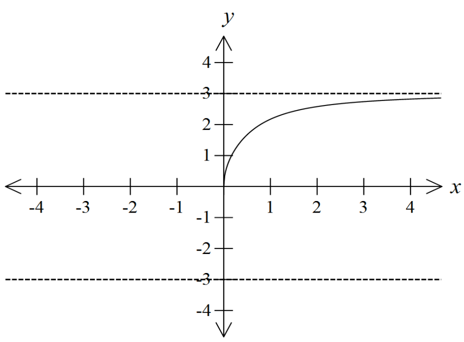
(B)



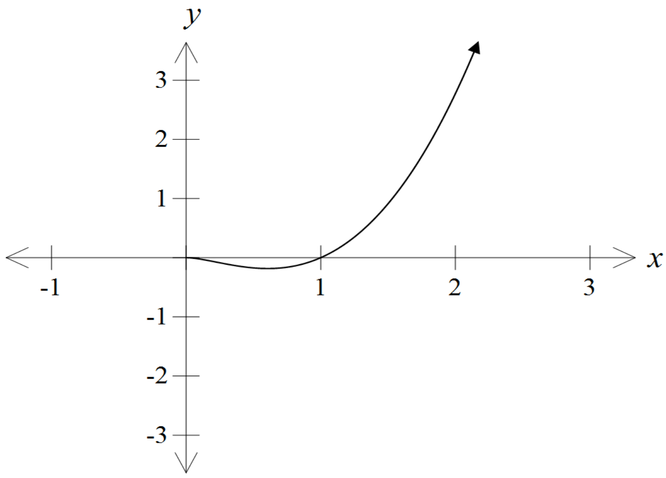
(C)



(D)

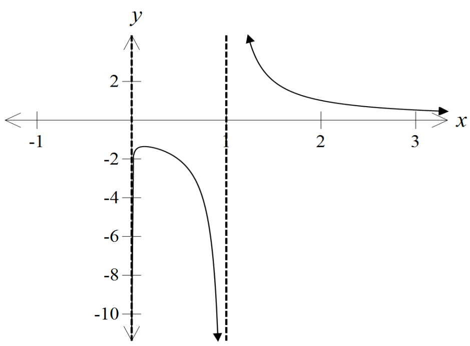


6 The diagram shows the graph of the function $y = f(x)$.

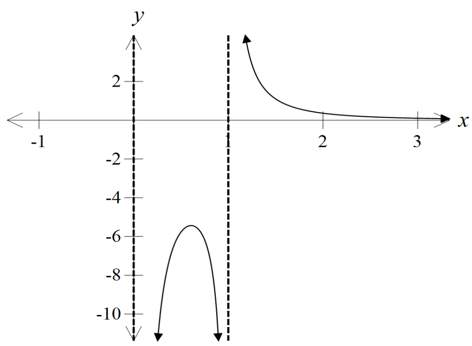


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

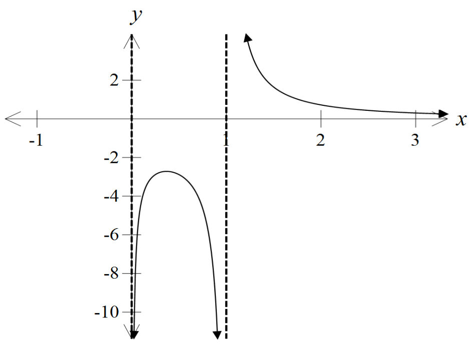
(A)



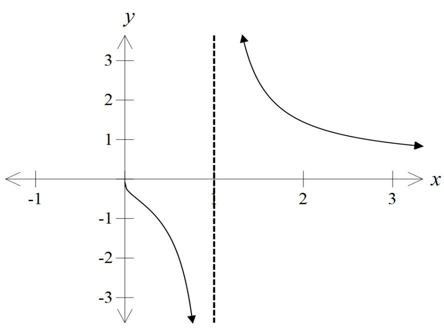
(B)



(C)



(D)



7 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ (B) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ (D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

8 Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

9 The equation $4x^3-27x+k=0$ has a double root.
What are the possible values of k?

(A) ± 4
(B) ± 9
(C) ± 27
(D) $\pm\frac{81}{2}$

10 Given that $(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$, then if $p(x) = (4x^2 + ax - 1)^2$, the value of a is:
(A) 1
(B) 2
(C) $\frac{1}{2}$
(D) 0

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Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new answer booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Start a new answer booklet** **Marks**

(a) Using the substitution $u = e^x + 1$ or otherwise, evaluate

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx. \qquad \mathbf{3}$$

(b) Find $\displaystyle\int \frac{1}{x \ln x}\,dx$. **1**

(c) (i) Find a, b, and c, such that

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}. \qquad \mathbf{2}$$

(ii) Find $\displaystyle\int \frac{16}{(x^2+4)(2-x)}\,dx$. **2**

(d) Using integration by parts ONLY, evaluate

$$\int_0^1 \sin^{-1} x\,dx. \qquad \mathbf{3}$$

(e) Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that :

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right). \qquad \mathbf{4}$$

Question 12 (15 marks) **Start a new answer booklet** **Marks**

(a) Given $z = \dfrac{\sqrt{3} + i}{1 + i}$,

(i) Find the argument and modulus of z. **2**

(ii) Find the smallest positive integer n such that z^n is real. **1**

(b) The complex number z moves such that $\text{Im}\left[\dfrac{1}{\bar{z} - i}\right] = 2$.

Show that the locus of z is a circle. **2**

(c) Sketch the region in the complex plane where the inequalities

$|z + 1 - i| < 2$ and $0 < \arg(z + 1 - i) < \dfrac{3\pi}{4}$ hold simultaneously. **3**

(d) Find the three different values of z for which

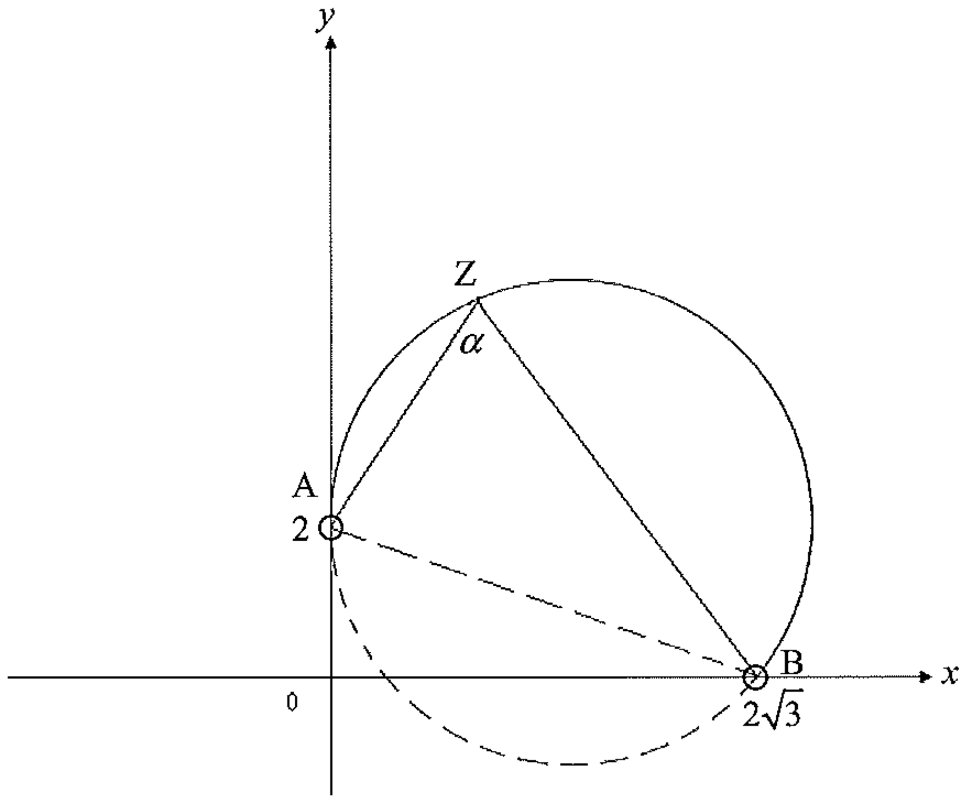
$z^3 = \dfrac{1 + i}{\sqrt{2}}$. **3**

Question 12 continues on the next page …….

(e) The locus of the complex number Z, moving in the complex plane such that

$arg(Z - 2\sqrt{3}) - arg(Z - 2i) = \dfrac{\pi}{3}$, is a part of a circle.

The angle between the lines from $2i$ to Z and then from $2\sqrt{3}$ to Z is α, as shown in the diagram below.



(i) Show that $\alpha = \dfrac{\pi}{3}$. **2**

(ii) Find the centre and the radius of the circle. **2**

Question 13 (15 marks) **Start a new answer booklet** **Marks**

(a) Consider the polynomial equation

$x^4 + ax^3 + bx^2 + cx + d = 0$

where $a, b, c,$ and d are all integers. Suppose the equation has a root of the form $x = ki$, where k is real, and $k \neq 0$.

(i) State why the conjugate $x = -ki$ is also a root. **1**

(ii) Show that $c = k^2 a$. **2**

(iii) Show that $c^2 + a^2 d = abc$. **2**

(iv) If $x = 2$ is also a root of the equation, and $b = 0$, show that d and c are both even. **2**

(b) (i) Solve $z^5 + 1 = 0$ by De Moivre's Theorem, leaving your solutions in modulus-argument form. **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$. **1**

(iii) Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = cis\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$. **3**

Hint: $z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$

(iv) Hence, find the exact value of $\sec\frac{3\pi}{5}$. **2**

Question 14 (15 marks) **Start a new answer booklet** **Marks**

(a) (i) Determine the real values of λ for which the equation

$$\frac{x^2}{4-\lambda}+\frac{y^2}{2-\lambda}=1 \text{ defines}$$

(α) an ellipse **1**

(β) a hyperbola **1**

(ii) Sketch the curve corresponding to the value $\lambda = 1$, indicating the positions of the foci and directrices and stating their coordinates and equations respectively. Also mark any axes intercepts on your diagram. **3**

(iii) Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached? **3**

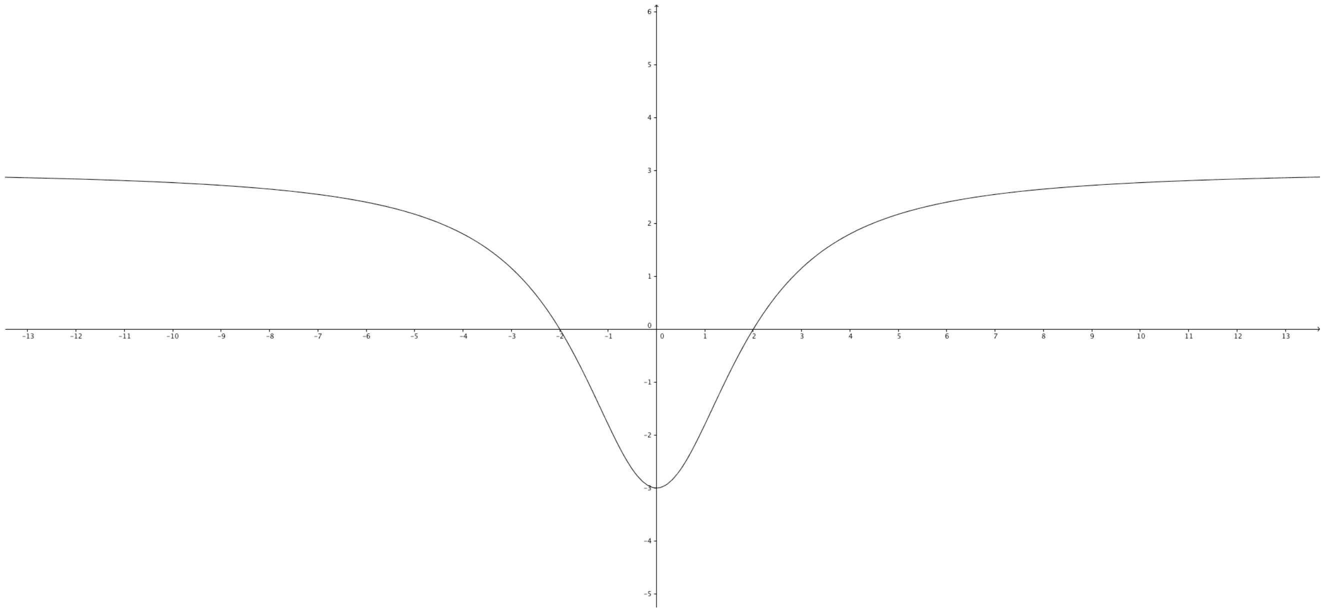
(b) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at $P(cp, \frac{c}{p})$ is $p^3x - py = c(p^4 - 1)$. **2**

(ii) The normal at $P(cp, \frac{c}{p})$ meets the hyperbola $xy = c^2$ again at $Q(cq, \frac{c}{q})$. Prove that $p^3 q = -1$. **2**

(iii) Hence, show that the locus of the midpoint R of PQ is given by $c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$. **3**

Question 15 (15 marks) **Start a new answer booklet** **Marks**

(a) Given below is the graph of $f(x) = 3 - \dfrac{24}{x^2 + 4}$.



Use the graph of $y = f(x)$ to sketch, on separate axes, the graphs of

(i) $y = \left[f(x) \right]^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

Each graph should be at least one – third of a page in size.

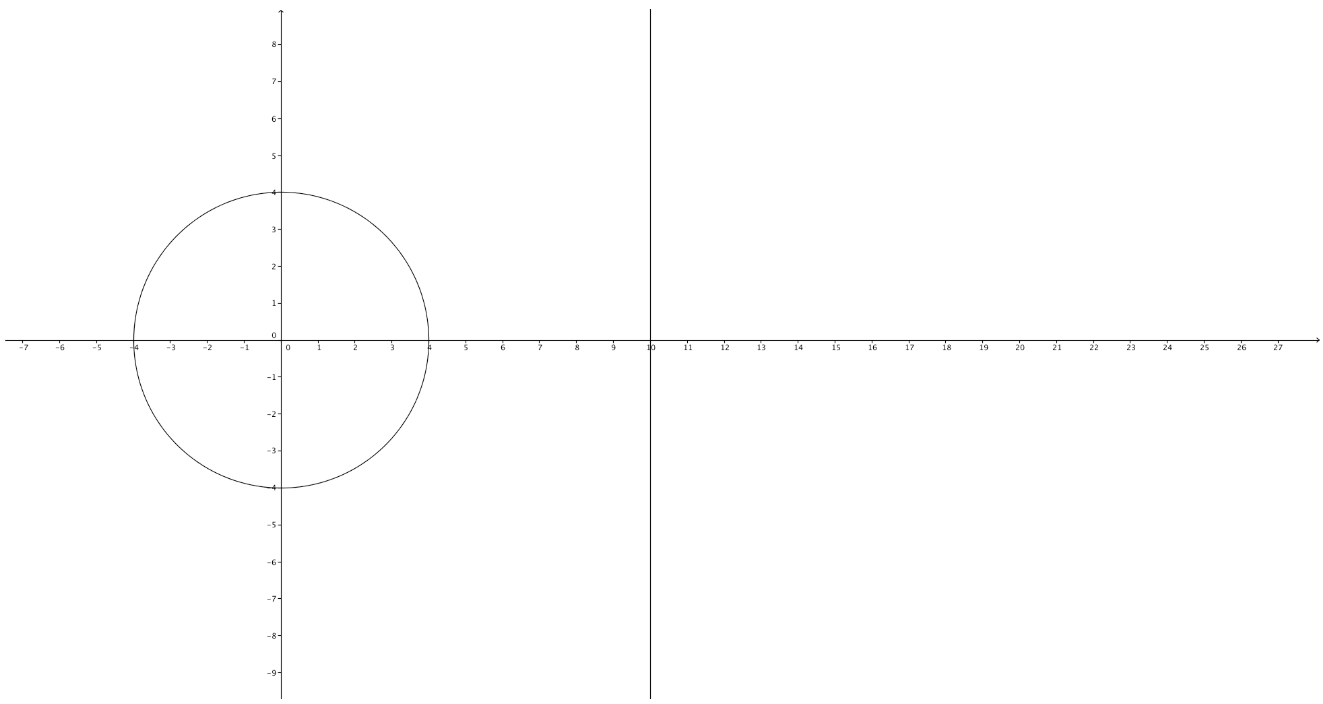
(b) Consider the curve that is defined by $4x^2 - 2xy + y^2 - 6x = 0$

(i) Show that $\dfrac{dy}{dx} = \dfrac{3 - 4x + y}{y - x}$ **2**

(ii) Find the coordinates of all points where the tangent is vertical. **2**

Question 15 continues on the next page …….

(c) A solid is formed by rotating the area enclosed by the curve $x^2 + y^2 = 16$ through one complete revolution about the line $x = 10$.



(i) Use the method of slicing to show that the volume of this solid is

$$V = 40\pi \int_{-4}^{4} \sqrt{16 - y^2}\, dy$$ **3**

(ii) Find the exact volume of the solid. **2**

Question 16 (15 marks) **Start a new answer booklet** **Marks**

(a) Let $f(x)=(1-\frac{x^2}{2})-\cos x$

(i) Show that $f(x)$ is an even function. **1**

(ii) Find expressions for $f'(x)$ and $f''(x)$. **2**

(iii) Deduce that $f'(x) \le 0$ for $x \ge 0$. **2**

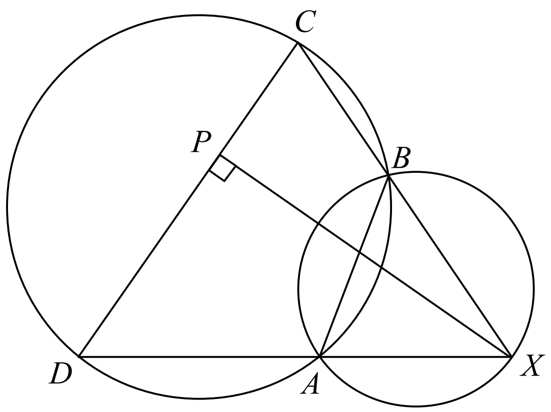
(iv) Hence, show that $\cos x \ge 1-\frac{x^2}{2}$. **2**

(b) (i) Use the principle of mathematical induction to prove that

$$(1+x)^n > 1+nx \text{ for } n>1 \text{ and } x>-1$$ **3**

(ii) Hence, deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n>1$. **1**

(c)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$.

(i) Prove that $\angle DBX = 90°$ **2**

(ii) Hence, or otherwise, prove that $AB = AP$. **2**

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\, dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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Year 12 Mathematics Extension 2

Section I - Answer Sheet

Student Number ______________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

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1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○