NORTH SYDNEY GIRLS HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

At the end of the examination, place your writing booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

NAME:______________________________ **TEACHER:**________________________

NUMBER:_________________________

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

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Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{6x}{\sqrt{1+x^2}}\,dx$. **2**

(b) By completing the square, or otherwise, evaluate $\displaystyle\int_{-1}^{5} \frac{dx}{\sqrt{32+4x-x^2}}$. **3**

(c) (i) Use integration by parts to find $\displaystyle\int (t-1)\ln t\,dt$. **3**

(ii) Using the substitution $t = 2x+1$, evaluate $\displaystyle\int_0^1 4x\ln(2x+1)\,dx$. **3**

(d) Use the substitution $t = \tan\frac{1}{2}\theta$ to show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta - 2\sin\theta + 3} = \frac{\pi}{4}$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w = 2 + i$.

(i) Find w^2 in the form $x + iy$. **1**

(ii) Find $\operatorname{Im}\left(\dfrac{1}{w}\right)$. **1**

(iii) Find the real numbers x and y such that $x + 3iy = w + 4i\overline{w}$. **2**

(b) (i) If $z = \cos\theta + i\sin\theta$ show that **2**

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(ii) Given further that $z + \dfrac{1}{z} = \sqrt{2}$, find the value of **2**

$$z^{10} + \frac{1}{z^{10}}.$$

(c) A circle C and a ray L have equations $\left| z - 2\sqrt{3} - i \right| = 4$ and $\arg(z + i) = \dfrac{\pi}{6}$ respectively.

(i) Show that:

(1) the circle C passes through the point where $z = -i$ **1**

(2) the ray L passes through the centre of C. **2**

(ii) Sketch C and L on the same Argand diagram. **2**

(iii) Shade on your sketch the region satisfying both **2**

$$\left| z - 2\sqrt{3} - i \right| \le 4 \text{ and } 0 \le \arg(z + i) \le \frac{\pi}{6}.$$

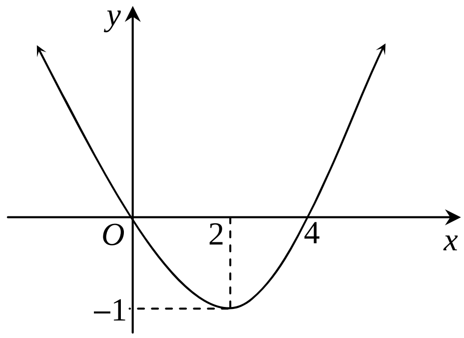
Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The sketch below shows the curve $y = f(x)$ where

$$f(x) = \frac{x(x-4)}{4}.$$

Without the use of calculus, draw sketches of the following, showing where necessary any intercepts, asymptotes and turning points

Use pages 14 and 15 to complete this question.



(i) $|y| = f(x)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \frac{x}{4}|x-4|$ **2**

(iv) $y = \tan^{-1}\left[f(x)\right]$. **2**

(b) (i) If $x \geq 0$, show that $\frac{x}{x^2+4} \leq \frac{1}{4}$. **2**

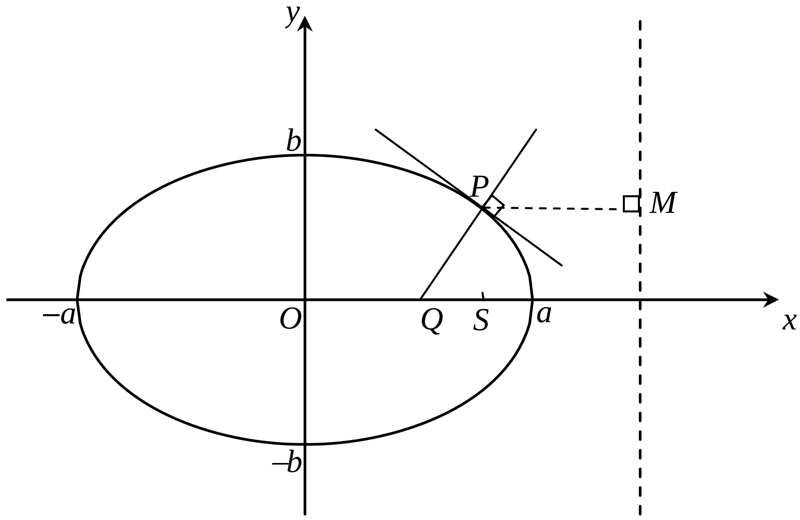
(ii) By integrating both sides of this inequality with respect to x between the limits $x = 0$ and $x = \alpha$, show that **2**

$$e^{\frac{1}{2}\alpha} \geq \frac{1}{4}\alpha^2 + 1 \text{ for } \alpha \geq 0.$$

(c) The region between the curve $y = 8x\sqrt{\sin 2x}$ and the x-axis for $0 \leq x \leq \frac{\pi}{2}$, is rotated about the x-axis. **4**
By using integration by parts twice, find the volume of the solid generated.
Leave your answer correct to 3 significant figures.

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$



(i) Write down the coordinates of the focus S and the equation of the associated directrix. **2**

(ii) Show that the equation of the normal to the ellipse at the point $P\left(x_1, y_1\right)$ is given by **2**

$$\frac{9x}{x_1}-\frac{5y}{y_1}=4.$$

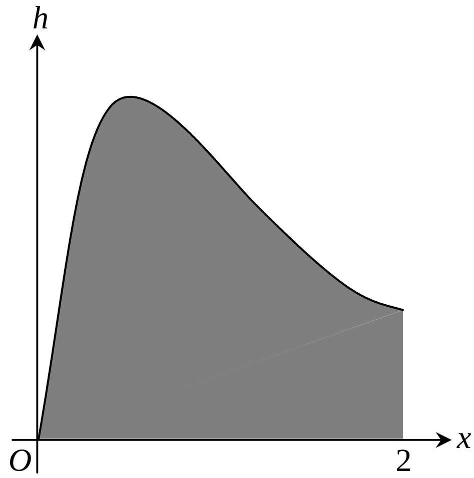
(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. Show that $QS=\frac{4}{9}PM$. **2**

(b) A curve has equation $x^3y+\cos\left(\pi y\right)=7$. **3**
Find the gradient of the curve at the point where $y = 1$.

Question 4 continues on page 7

(c) NSGHS is planning to construct an artwork in the Senior Lawn from pre-made panels. One side of each panel needs to be painted.

To determine the amount of paint needed, the area of one side of each panel needs to be calculated.
Each panel is 2 m wide. An artist's sketch of a panel is given below.



Rekrap examines the artist's sketch and decides that the height of each panel can be modeled by the following function:

$$h(x) = \frac{10x}{(x^2+1)(3x+1)}, \quad 0 \le x \le 2$$

(i) Given that $\dfrac{10x}{(x^2+1)(3x+1)}$ can be written in the form $\dfrac{x+A}{x^2+1} + \dfrac{B}{3x+1}$, find the values of A and B. **3**

(ii) Hence, find the area of one of the panels correct to 2 decimal places. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The cubic equation $z^3 + iz^2 + 3z - i = 0$ has roots α, β, and γ.

Write down a polynomial equation that has roots

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$, and $\dfrac{1}{\gamma}$. **1**

(ii) α^2, β^2, and γ^2. **2**

(b) A curve has equation

$$y = \frac{x^2}{(x-1)(x-5)}.$$

(i) Show that, if the curve intersects the line $y = k$, then the x-coordinates of the points of intersection must satisfy the equation **1**

$$(k-1)x^2 - 6kx + 5k = 0.$$

(ii) Show that if the equation in (i) has equal roots then **2**

$$k(4k+5) = 0.$$

(iii) Hence find the coordinates of the two stationary points on the curve. **3**

(iv) Sketch the curve showing intercepts, asymptotes and stationary points. **3**

(c) (i) Differentiate $x(1+x)^n$. **1**

(ii) Hence, show that **2**

$$\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} - 4\binom{n}{3} + \ldots + (-1)^n (n+1)\binom{n}{n} = 0.$$

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.
It is given that $\alpha = 2 + 2\sqrt{3}i$.

(i) Write down another root, β, of the equation. **1**

(ii) Find the third root, γ. **2**

(iii) Find the values of p and q. **2**

(iv) By expressing α in modulus-argument form, show that **2**

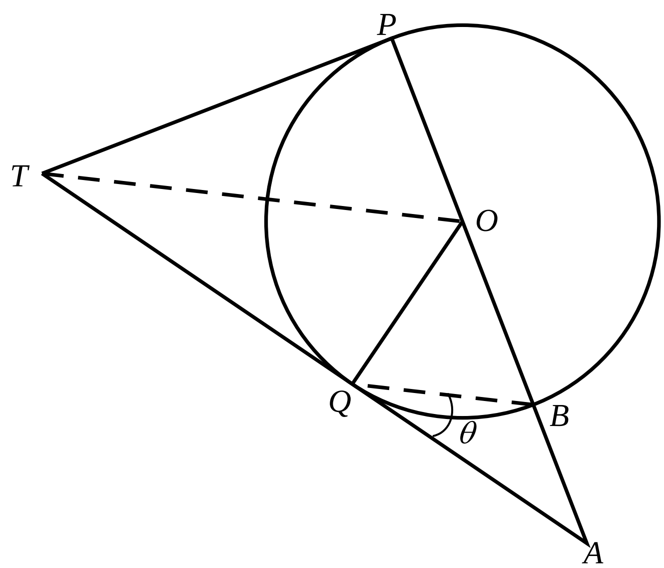
$$\left(2 + 2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right).$$

(v) Hence, show that **2**

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n.$$

where n is an integer.

(b) From an external point T, tangents are drawn to a circle with centre O, touching the circle at P and Q. Angle PTQ is acute.
The diameter PB produced meets the tangent TQ at A. Let $\angle AQB = \theta$.



(i) Copy the diagram above into your answer booklet.

(ii) Prove that $\angle PTQ = 2\theta$. **2**

(iii) Prove that ΔPBQ and ΔTOQ are similar. **2**

(iv) Hence show that $BQ \times OT = 2 \times OP^2$. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola $xy = c^2$ and the distinct points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$

(i) Show that the equation of the tangent at $P\left(cp, \dfrac{c}{p}\right)$, where $p \neq 0$ is **2**

$$x + p^2 y = 2cp.$$

(ii) Show that the tangents at P and Q intersect at **2**

$$M\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right).$$

(iii) Show that if $pq = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin. **2**

(b) For each integer $n \geq 0$ let

$$I_n = \int_0^1 x\left(x^2 - 1\right)^n \, dx$$

(i) Using the method of integration by parts, show that for $n \geq 1$ **3**

$$I_n = -\frac{n}{n+1} I_{n-1}.$$

(ii) Hence, or otherwise, show that for $n \geq 0$ **2**

$$I_n = \frac{(-1)^n}{2(n+1)}.$$

(iii) Show that for n odd that $I_n < I_{n+2}$ for all $n \geq 0$. **2**

(iv) Deduce that $-\dfrac{1}{4n} < I_{2n+1} < -\dfrac{1}{4(n+2)}$. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2 \text{ and } u_{k+1} = 2u_k + 1$$

(i) Prove by induction that, for all integers $n \geq 1$, **3**

$$u_n = 3 \times 2^{n-1} - 1.$$

(ii) Show that **2**

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2).$$

(b) (i) If $a > 0$, $b > 0$ and $c > 0$, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

(ii) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and **3**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{27}{abc}.$$

(c) (i) Show that if n is any even positive integer, then **2**

$$(1+x)^n + (1-x)^n = 2\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} x^{2k}.$$

(ii) An alphabet consists of the three letters A, B and C.

(1) Show that the number of words of 4 letters containing exactly 2 As is **1**

$$\binom{4}{2} \times 2^2.$$

(2) Hence, or otherwise, show that if n is an even positive integer, then the number of words of n letters with zero or an even number of As is given by **2**

$$\frac{1}{2}\left(3^n + 1\right).$$

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

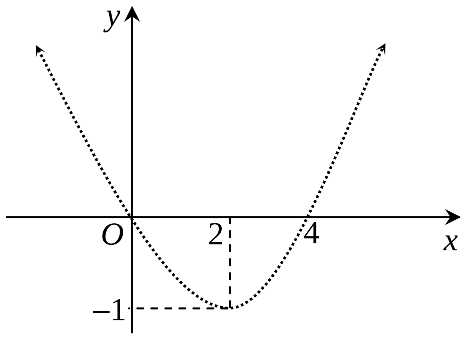
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

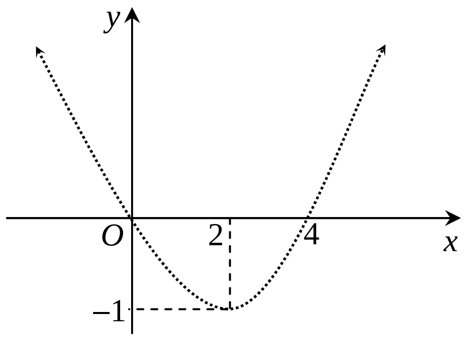
NSGHS 2011 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 3 (a)

Number________________________________ **Teacher:________________________**

(i)



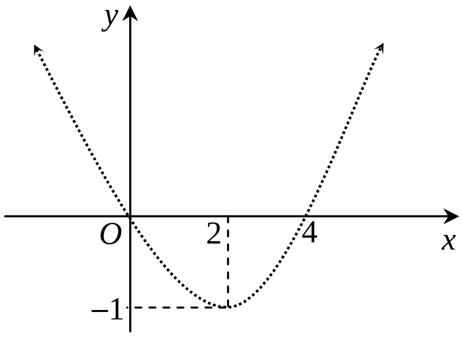
(ii)



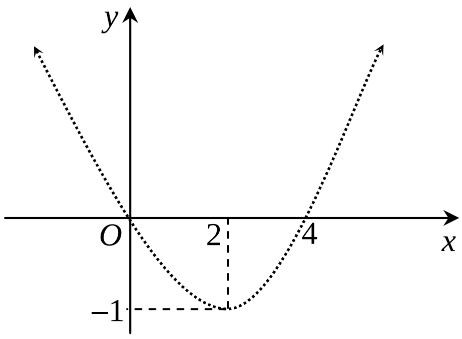
Turn over for parts (iii) and (iv)

Answer Sheet for Question 3 (a) continued

(iii)



(iv)



Now place this sheet <u>INSIDE</u> your booklet for Question 3