The Scots College

HSC Mathematics Extension 2

Trial Examination

12^{th} August 2011

Name:_______________________

General Instructions

- Working time : 3 hours + 5 minutes reading time.
- Write using blue or black pen
- Board approved calculators may be used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached
- Answer each question on a SEPARATE answer booklet

TOTAL MARKS: 120

Attempt Questions 1 - 8
All questions are of equal value

WEIGHTING: 40 %

Question 1 (Marks 15) Use a SEPARATE writing booklet.

a) Evaluate [2]

$$\int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$$

b) Find [3]

$$\int \frac{dx}{\sqrt{x\,(2-x)}}$$

using the substitution $\sqrt{\frac{x}{2}} = \sin\theta$

c) Show that [4]

$$\int_1^3 \frac{2x^2-3x+11}{(x+1)(x^2-2x+5)}\, dx = 2\, ln2 + \frac{\pi}{8}$$

d) Find [3]

$$\int \, x \ln(x+1)\, dx$$

e) Find [3]

$$\int \frac{dx}{1 + \text{sinx} + \text{cosx}}$$

Question 2 (Marks 15) Use a SEPARATE writing booklet.

a) i) Find the square root of $-5-12i$. [2]

ii) Hence solve $z^2 - iz + 1 + 3i = 0$, expressing your answer in the form $a + ib$, where *a* and *b* are real numbers. [2]

b) i) Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in the modulus – argument form. [2]

ii) Hence express $\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^{10}$ in the form $x + iy$, where *x* and *y* are both real. [2]

c) i) Sketch on the Argand diagram the locus of the complex number *z*, which satisfies the condition [2]

$$\arg\left(\frac{z-2}{z-2i}\right) = \frac{\pi}{2}$$

(ii) Hence, or otherwise, find the complex number *z* (in the form *a* + *ib*, where *a* and *b* are both real) which has the maximum value of $|z|$. [1]

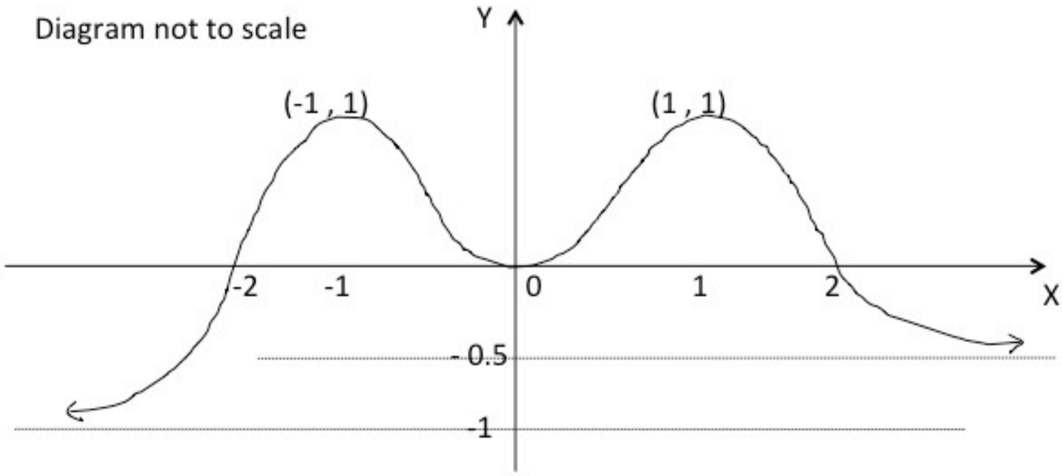
d) Let $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

i) Show that $1 - z + z^2 - z^3 + z^4 = 0$ [2]

ii) Show that $(1-z)(1+z^2)(1-z^3)(1+z^4) = 1$ [2]

Question 3 (Marks 15) Use a SEPARATE writing booklet.

a) The graph of $y = f(x)$ is given below. [10]



Using the graph of $y = f(x)$, sketch on separate axes, the graphs of

i. $|y| = |f(x)|$

ii. $y^2 = f(x)$

iii. $y = \frac{1}{f(x)}$

iv. $y = e^{f(x)}$

v. $y = sin^{-1} f(x)$

b) (i) Show that if $I_n = \int_0^1 x^n \sqrt{1 - x^2}\, dx$, then [3]

$$I_n = \frac{n-1}{n+2} I_{n-2}$$

(ii) Hence evaluate $\int_0^1 x^4 \sqrt{1 - x^2}\, dx$ [2]

Question 4 (Marks 15) Use a SEPARATE writing booklet.

a) A hyperbola has the asymptotes $y = x$ and $y = -x$, and it passes through the point (5,4). Find [7]

i. the equation of the hyperbola

ii. its eccentricity

iii. the length of the principal axis

iv. the coordinates of the foci

v. equation of the directrices

vi. the length of the latus rectum

b) The point $P\left(cp, \frac{c}{p}\right)$, lies on the hyperbola $xy = c^2$. The tangent at P meets the x – axis at A and the y – axis at B . The normal to the hyperbola at P meets the line $y = x$ at the point C . [8]

i. Show that the equation of the tangent at P is $x + p^2y = 2cp$.

ii. Find the coordinates of A and B .

iii. Find the equation of the normal at P .

iv. Show that the x – coordinate of the point C is given by $x = \frac{c}{p}(p^2 + 1)$.

v. Prove that $\triangle ABC$ is an isosceles triangle.

Question 5 (Marks 15) Use a SEPARATE writing booklet.

a) $P\ (a\cos\theta,\ b\sin\theta\ )$ is a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

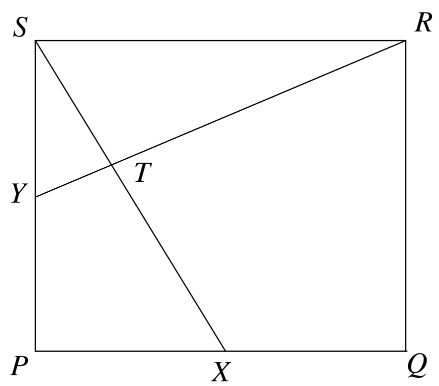
i. Show that the equation of the tangent to the ellipse at P is $bx\cos\theta + ay\sin\theta = ab$. [2]

ii. Deduce the equation of the normal to the ellipse at P . [2]

iii. Find the coordinates of X and Y, the points where the tangent and the normal respectively, meet the y – axis. [2]

iv. Show that the circle with XY as the diameter, passes through the foci of the ellipse.. [3]

b) *PQRS* is a square. *X* and *Y* are mid-points of *PQ* and *PS* respectively. *SX* and *RY* intersect at the point *T*. [6]



i. Prove that *QRTX* is a cyclic quadrilateral.

ii. Hence prove that $QT = QR$

Question 6 (Marks 15) Use a SEPARATE writing booklet.

a) A solid is formed by rotating the circle $x^2 - 2ax + y^2 = 0$ about the line $x = 3a$. [5]

Find the volume of the solid generated by taking slices perpendicular to the axis of rotation.

b) The base of a certain solid is the region between the curve $y = \frac{x^3}{4}$, $0 \leq x \leq 2$, and the line $y = x$. [6]

Each plane section of the solid perpendicular to the x-axis is a parabola whose chord lies on the base of the solid, with one end point *A* on the line $y = x$ and the other end point *B* on the curve $y = \frac{x^3}{4}$. The axis of the parabola is vertical and passing through the mid-point of *AB* and the maximum height of the parabola, from the base, is equal to the length of the chord *AB* .

By first finding the area of a slice taken perpendicular to the *x* – axis, find the volume of the solid.

c) A sequence $u_1, u_2, u_3, \ldots\ldots\ldots$ is defined by the relation [4]

$u_n = u_{n-1} + 6u_{n-2}$, for $n \geq 3$.

Given that $u_1 = 1$ and $u_2 = -12$, prove by using mathematical induction

$u_n = -6[(-2)^{n-2} + 3^{n-2}]$, for all positive integers *n* .

Question 7 (Marks 15) Use a SEPARATE writing booklet.

a) i. By using De Moivre's Theorem or otherwise, prove that [3]

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

ii. Using part (i) solve the equation [4]

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

and hence find the value of

$$\tan\frac{\pi}{16} \times \tan\frac{3\pi}{16} \times \tan\frac{5\pi}{16} \times \tan\frac{7\pi}{16}$$

b) It is given that the product of two of the roots of the equation $x^4 + x^3 - 16x^2 - 4x + 48 = 0$, is equal to 6. [4]

Show that the equation can be written in the form $(x^2 + ax + b)(x^2 + cx + d) = 0$, where *a*, *b*, *c* and *d* are integers.

Hence or otherwise solve the equation.

c) i. Find all the values of *m* for which the polynomial $3x^4 - 4x^3 + m = 0$ has no real roots. [4]

ii. Determine the real roots of the polynomial when $m = 1$

Question 8 (Marks 15) Use a SEPARATE writing booklet.

a) A particle P of mass m kg is projected vertically upwards from the ground, with an initial velocity of u m/s, in a medium of resistance mkv^2, where k is a positive constant and v is the velocity of the particle. [8]

i. Show that the maximum height H, from the ground, attained by the particle P is given by $H = \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$, where g is the acceleration due to gravity.

ii. At the same time that P is projected upwards, another particle, Q, of equal mass, initially at rest, is allowed to fall downwards in the same medium, from a height of H metres from the ground, along the same vertical path as P. Show that at the time of collision of P and Q,

$$\frac{1}{v_2^2} - \frac{1}{v_1^2} = \frac{1}{V^2},$$

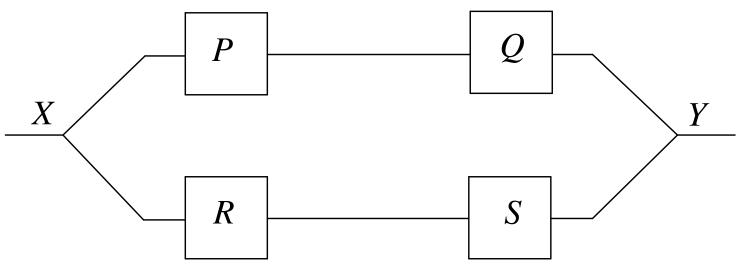
where v_1 and v_2 are the velocities of particles P and Q respectively, at the time of collision, and $V = \sqrt{\frac{g}{k}}$.

b) Find the stationary points, stating their nature, for the curve [3]

$$x^2 + y^2 = xy + 3$$

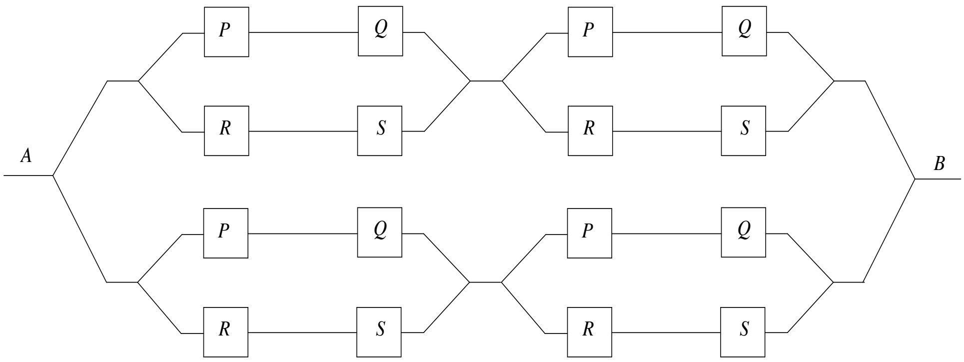
c) i. An electrical circuit has four bulbs P, Q, R and S placed as shown in the diagram. The probability of each bulb being defective, independently, is given by p. Current can flow from X to Y through either or both of the branches of the electrical circuit. However, no current will flow through a branch that has at least one defective bulb. [4]

Show that the probability that the current *does not* flow from X to Y is $(2p - p^2)^2$



Question 8 continued........

ii. In the household there are four such circuits in one connection. Find the probability that current does not flow from A to B .



End of Assessment

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$