Solved

Department of Mathematics
University of Toronto

TUESDAY, March 10, 2009 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1074	T0501D	W3D	RW142
T0101B	M9B	SS2105	T0601A	R4A	LM 157
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0101D	M9D	LM 158	T0701A	F2A	RW 229
T0201A	M3A	RW 229	T0701B	F2B	SS2111
T0201B	M3B	LM 157	T0701C	F2C	SS2128
T0201C	M3C	RW 142	T0801A	F3A	LM 155
T0201D	M3D	UC 52	T0801B	F3B	LM 123
T0301A	T3A	RW 143	T5101A	M5A	MP 134
T0301B	T3B	MP 134	T5101B	M5B	SS2111
T0401A	W9A	SS1074	T5101C	M5C	MP 118
T0401B	W9B	SS1086	T5101D	M5D	RW 143
T0401C	W9C	LM 158	T5201A	M6A	LM 162
T0501A	W3A	SS1086			
T0501B	W3B	MS2173			
T0501C	W3C	UC 256			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

$\lim_{x\to 1} \frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{4}}-1}$ is $= \lim_{x\to 1} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{4}x^{-\frac{1}{4}}} = 2$ since $\frac{0}{0}$

A. 0

B. 1

[✓ C.] 2

D. −1

E. undefined

or $= \lim_{x\to 1} \frac{(x^{\frac{1}{2}}-1)(x^{\frac{1}{4}}+1)}{(x^{\frac{1}{4}}-1)(x^{\frac{1}{4}}+1)} = \lim_{x\to 1} \frac{\cancel{(x^{\frac{1}{2}}-1)}}{\cancel{x^{\frac{1}{2}}-1}}(x^{\frac{1}{4}}+1)$

$= 2$

2. *[4 marks]*

$\lim_{x\to+\infty} (1+e^x)^{\frac{3}{x}} =$

A. 1

B. $1+e^3$

C. 3

[✓ D.] e^3

E. $+\infty$

Let $y = (1+e^x)^{\frac{3}{x}}$

$\ln y = \frac{3\ln(1+e^x)}{x}$

$\frac{\infty}{\infty}$

so $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{3e^x}{1+e^x}}{1}$

$= \lim_{x\to\infty} \frac{3e^x}{e^x+1} = \lim_{x\to\infty} \frac{3}{1+e^{-x}} = 3$

or, since still $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{3e^x}{e^x} = 3$

$\lim_{x\to\infty} \ln y = 3$

so $\lim_{x\to+\infty} y = \lim_{x\to\infty} e^{\ln y} = e^3$

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3. *[4 marks]*

Suppose that $y = f(x)$ satisfies the following:

i) $f(x)$ is increasing for all values of x

ii) as $x \to +\infty$ $f(x) \to 1^-$

as $x \to -\infty$ $f(x) \to -1^+$

iii) $f''(x)$ is defined and continuous for all values of x.

Which of the following statements is false?

A. $f(x) = 0$ for exactly one value of x – True: f is cont. since f'' is and must increase from -1 to + 1, passing through 0 once only.

B. $f'(x) \geq 0$ for all values of x - True - f increasing.

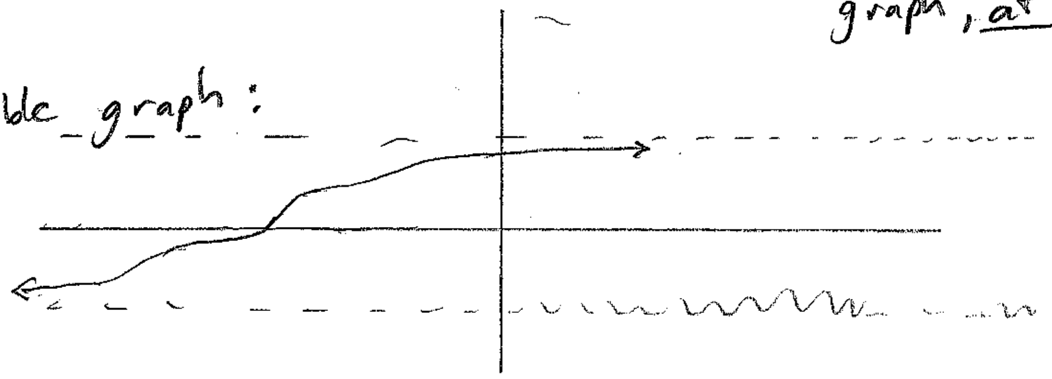
C. f has exactly two horizontal asymptotes - True - that's what ii) says.

D. f has at least one inflection point – True - f has to be concave up near $-\infty$ and concave down near $+\infty$, and since f'' is cont, inflection must change on the graph, at least once.

[✓ E.] f could have a vertical tangent

False - f is cont. since f'' is cont.

Possible graph :



4. *[4 marks]*

If $f(x) = \int_0^x \sqrt{1+t^3}\,dt$ then $f'(2) =$

[✓ A.] 3

B. 2

C. 1

D. 0

E. -1

$f'(x) = \sqrt{1+x^3}$

$f'(2) = \sqrt{1+8} = 3$

5. *[4 marks]*

$\int_1^2 xe^{-x^2}\,dx =$

A. $\frac{2}{e^4} - \frac{1}{e}$

B. $\frac{1}{2}(1 - \frac{1}{e})$

C. $\frac{1}{2}(\frac{1}{e} - \frac{1}{2e^4})$

[✓ D.] $\frac{1}{2}(\frac{1}{e} - \frac{1}{e^4})$

E. $(\frac{1}{e} - \frac{2}{e^4})$

Let $u = x^2$ $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

When $x=1, u=1$

$x=2, u=4$

$$\frac{1}{2}\int_1^4 e^{-u}\,du$$

$$= -\frac{1}{2}e^{-u}\Big|_1^4$$

$$= \frac{1}{2}(e^{-1} - e^{-4})$$

6. *[4 marks]*

The demand equation for a product is $p = (q-6)^2$ and the supply equation is $p = q^2 - q + 3$. The consumers' surplus under market equilibrium is

A. −36

B. 12

C. 108

D. 54

[✓ E.] 36

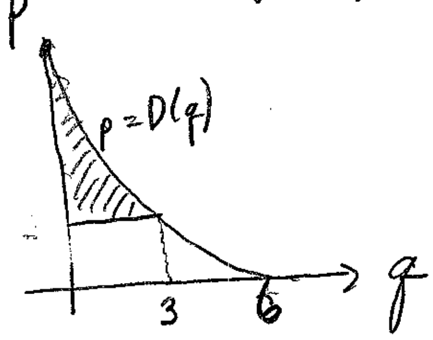
Equilibrium at $(q-6)^2 = q^2 - q + 3$

$q^2 - 12q + 36 = q^2 - q + 3$

$33 = 11q$

$q_0 = 3$

$p_0 = 9$



$$CS = \int_0^{q_0} [D(q) - p_0]\,dq = \int_6^3 [(q-6)^2 - 9]\,dq = \int_0^3 (q^2 - 12q + 27)\,dq$$

$$= \left[\frac{q^3}{3} - 6q^2 + 27q\right]_0^3 = 9 - 54 + 81 = 36$$

7. *[4 marks]*

$\int (\ln x)^2\,dx$ is equal to

[✓ A.] $x \ln x(\ln x - 2) + 2x + C$

B. $(\ln x)^3/3 + C$

C. $2e^{x-\ln x} + C$

D. $2x \ln x + C$

E. $\dfrac{2\ln x}{x} + C$

By parts:

$u = (\ln x)^2 \qquad dv = dx$

$du = \dfrac{2\ln x}{x} \qquad v = x$

$\int = x(\ln x)^2 - 2\int \ln x\, dx$

$u = \ln x \qquad dv = dx$

$du = \dfrac{dx}{x} \qquad v = x$

$\int = x(\ln x)^2 - 2[x\ln x - \int dx]$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

$= x\ln x(\ln x - 2) + 2x + C$

Note: Differentiating all the choices will also get you there.

8. *[4 marks]*

If A and B are real numbers such that $\dfrac{A}{x-1} + \dfrac{B}{x+1} = \dfrac{x+2}{x^2-1}$ for all x such that $x^2 \neq 1$, then $A =$

A. $\frac{5}{2}$

B. $-\frac{1}{2}$

[✓ C.] $\frac{3}{2}$

D. $\frac{1}{2}$

E. $-\frac{3}{2}$

$A(x+1) + B(x-1) = x+2$

if $x=1$, $2A = 3$ and $A = \frac{3}{2}$

or: $(A+B)x + (A-B) = x+2$

$A+B=1$

$A-B=2$

$2A = 3$

$A = \frac{3}{2}$

9. *[4 marks]*

Cash flows into an account at a constant rate of 365 dollars per year, beginning now. If the account earns interest at 5% compounded continuously, then in 10 years it will have

A. \$4671.55

[✓ B.] \$4735.67

C. \$4590.93

D. \$4925.36

E. \$4462.08

$$F.V. = \int_0^{10} 365\, e^{.05(10-t)}\, dt$$

$$= 365 e^{.5} \int_0^{10} e^{-.05t}\, dt$$

$$= \frac{365 e^{.5}}{-.05}\, e^{-.05t} \Big|_0^{10}$$

$$= 7300 e^{.5} (1 - e^{-.5})$$

$$= 7300 (e^{.5} - 1)$$

$$= 4735.6652 \ldots$$

10. *[4 marks]*

The average value of $f(x) = 2^x$ on the interval $[0,2]$ is

A. 1.44

B. 1.50

[✓ C.] 2.16

D. 2.50

E. 2.88

$$\frac{1}{2-0} \int_0^2 2^x\, dx$$

$$= \frac{1}{2}\, \frac{2^x}{\ln 2} \Big|_0^2$$

$$= \frac{1}{2\ln 2} (4-1) = \frac{3}{2\ln 2} \approx 2.164 \ldots$$

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PART B. Written-Answer Questions

1. *[16 marks]*

Given: $f(x) = \frac{e^x}{x^2}$

[4] (a) Find all horizontal and vertical aysmptotes of f (justify your answer)

$\boxed{\text{V.A. at } x=0 \text{, since } e^0 = 1}$

$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$ so $\boxed{\text{no H.A. at } +\infty}$

$\lim_{x\to-\infty} \frac{e^x}{x^2} = 0$ since $e^x \to 0$ and $x^2 \to \infty$

[L'Hôpital's Rule does not apply here.]

$\boxed{y=0 \text{ is a H.A. at } -\infty}$

[4] (b) Given that $f'(x) = \frac{e^x(x-2)}{x^3}$ find when f is increasing and decreasing and find all relative extrema.

Interval	f'	f	
$(-\infty, 0)$	+	increasing	(x any neg. number will do)
$(0, 2)$	−	decreasing	($x=1$ will do)
$(2, \infty)$	+	increasing	($x=3$ will do)

There is a $\boxed{\text{local min at } x=2}$ by the first deriv. test.

Or, using $f''(2) = \frac{e^2}{8} > 0$ (see (c))

the second deriv. test says the same.

[3] (c) Given that $f''(x) = \frac{e^x[(x-2)^2+2]}{x^4}$ find when f is concave upwards and downwards and find all inflection points.

Since $f'' > 0$ wherever it is defined, the curve is $\boxed{\text{always}}$ $\boxed{\text{concave up and has no pts. of inflection}}$

This question continues on Page 8

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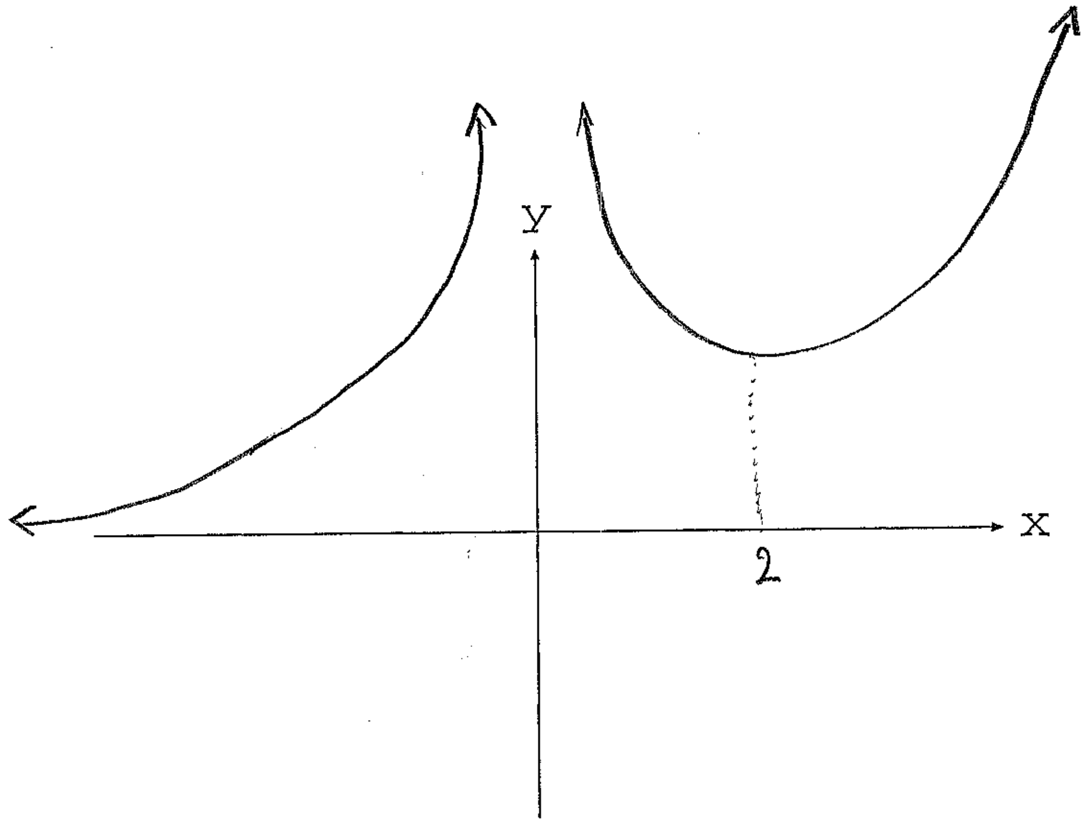
[5] B1(d)

Draw a clear sketch of $y = f(x)$ on the axes below.



2. *[14 marks]*

For a manufacturer to produce q units of a product, the **average** cost is

$$\bar{c}(q) = \frac{q^2}{3} - \frac{19q}{2} + 60 + \frac{400}{q}$$

[9] (a) If production is limited to at most 20 units find the number of units to be produced to minimize the **total** cost.

$$c(q) = q\,\bar{c}(q) = \frac{q^3}{3} - \frac{19q^2}{2} + 60q + 400$$

$$c'(q) = q^2 - 19q + 60$$

$$= (q-15)(q-4)$$

crit. pts. at $q=4$ and $q=15$.

c is a cont. fcn. on the interval $[0,20]$ so must have a minimum at $q = 0, 4, 15,$ or 20.

$c(0) = 400$

$c(4) \approx 509$

$c(15) \approx 287.5$ $\boxed{q=15 \text{ gives min}}$

$c(20) \approx 467$

[5] (b) If production is limited to at most 14 units find the number of units to be produced to minimize the **total** cost.

Everything is the same as in (a) but the interval is only $[0,14]$

$c(0) = 400$

$c(4) = 509$ (a max actually)

$c(14) \approx 293$ $\boxed{q = 14 \text{ gives } \cancel{\text{the}} \text{ min}}$

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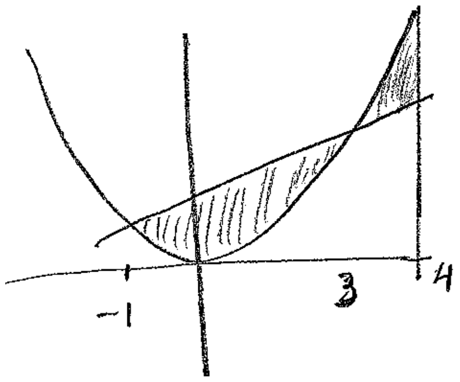
3. *[15 marks]*

Find the total area of all region(s) in the xy-plane which are bounded by the graph of $y = x^2$ and the lines $x = 4$ and $y = 2x + 3$. A rough sketch may be useful.



$y = 2x+3$ and $y = x^2$ intersect when $x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = -1$ and $x = 3$

The area of the shaded region is

$$\int_{-1}^{3} [(2x+3) - x^2]\,dx + \int_{3}^{4} [x^2 - (2x+3)]\,dx$$

$$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} + \left[\frac{x^3}{3} - x^2 - 3x\right]_{3}^{4}$$

$$= [(9+9-9) - (1-3+\tfrac{1}{3})] + [(\tfrac{64}{3} - 16 - 12) - (9-9-9)]$$

$$= [9 + \tfrac{5}{3}] + [\tfrac{64}{3} - 28 + 9]$$

$$= -10 + \frac{69}{3} = \boxed{13}$$

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4. *[15 marks]*

[8] (a) If $\frac{dy}{dx} = xy + x$ and $y = 0$ when $x = 1$ then, showing all your work, find y when $x = 3$.

$$\frac{dy}{dx} = x(y+1) \qquad \int \frac{dy}{y+1} = \int x\,dx$$

$$\ln(y+1) = \frac{x^2}{2} + C \qquad y+1 = Ae^{\frac{x^2}{2}}$$

$y=0$ when $x=1$, so $1 = Ae^{\frac{1}{2}}$ so $A = e^{-\frac{1}{2}}$

and $y = e^{\frac{x^2}{2} - \frac{1}{2}} - 1$

When $\boxed{x=3, \quad y = e^4 - 1}$

or $\boxed{y \approx 54.6}$

[7] (b) Find the following integral or show that it diverges

$$\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{7-x}} = \lim_{R \to \infty} \int_{-R}^{-1} (7-x)^{-\frac{1}{3}} dx$$

$$= \lim_{R \to \infty} \left[\frac{3}{2} (7-x)^{\frac{2}{3}} (-1) \right]_{-R}^{-1}$$

$$= \lim_{R \to \infty} \frac{3}{2} \left[(7+R)^{2/3} - 8^{2/3} \right], \text{ But } (7+R)^{\frac{2}{3}} \to \infty$$

so $\boxed{\text{the integral diverges}}$