Solved

Department of Mathematics
University of Toronto

TUESDAY, March 2, 2010 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1072	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	SS2127
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0201A	M3A	SS1086	T0701A	F2A	MP 118
T0201B	M3B	RW 142	T0701B	F2B	SS2105
T0201C	M3C	LM 157	T0701C	F2C	LM 155
T0201D	M3D	SS2110	T0701D	F2D	RW 143
T0301A	T3A	SS2105	T0801A	F3A	SS2111
T0301B	T3B	SS1074	T0801B	F3B	SS1088
T0301C	T3C	SS1084	T0801C	F3C	RW 143
T0401A	W9A	SS1072	T5101A	M5A	SS2110
T0401B	W9B	SS1088	T5101B	M5B	SS2128
T0501A	W3A	BA2195	T5101C	M5C	RW 229
T0501B	W3B	BA3004	T5101D	M5D	LM 123
T0501C	W3C	GB 404	T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1 /16	
B2 /15	
B3 /12	
B4 /17	
TOTAL/100	

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PART A. Multiple Choice

1. *[4 marks]*

If $y = \dfrac{5^x (x^2+2)^4}{(x^4+1)^3}$ then at $x = 0$, $\dfrac{dy}{dx} =$

A. $16 \ln 5$

B. 16

C. $\ln 5$

D. $\dfrac{\ln 5}{16}$

E. $\dfrac{16}{\ln 5}$

$$\ln y = x \ln 5 + 4 \ln(x^2+2) - 3 \ln(x^4+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 5 + \frac{8x}{x^2+2} - \frac{12x^3}{x^4+1}$$

At $x=0$, $y = 2^4 = 16$

so $\dfrac{1}{16}\dfrac{dy}{dx} = \ln 5$

$$\frac{dy}{dx} = 16 \ln 5$$

2. *[4 marks]*

$\ln(x+y) - x^2 y = 1$ defines y implicitly as a function of x near $x = 0$, $y = e$.
At $x = 0$, $y = e$, $\dfrac{dy}{dx} =$

A. 1

B. 0

C. -1

D. $e^2 - 1$

E. $\dfrac{1}{e} - 1$

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right) - 2xy - x^2\frac{dy}{dx} = 0$$

At $x=0$, $y=e$

$\dfrac{1}{e}\left(1+\dfrac{dy}{dx}\right) = 0$ $\quad \dfrac{dy}{dx} = -1$

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3. *[4 marks]*

$x + xy + y = 5$ defines y implicitly as a function of x near the point $x = 2$, $y = 1$.

At $(2, 1)$, $\dfrac{d^2y}{dx^2} =$

$1 + y + xy' + y' = 0$

At $x = 2, y = 1$

$1 + 1 + 2y' + y' = 0$

$y' = -\frac{2}{3}$

$y' + y' + xy'' + y'' = 0$

$2y' + (1+x)y'' = 0$

$-\frac{4}{3} + 3y'' = 0$

$y'' = \frac{4}{9}$

A. $-\frac{1}{9}$

B. $\frac{4}{9}$

C. $-\frac{2}{3}$

D. -1

E. $\frac{19}{9}$

4. *[4 marks]*

$\displaystyle\lim_{x\to 3} \frac{2\ln(x-2) - 2x + 6}{(x-3)^2} =$

$\frac{0}{0}$

$\displaystyle\lim_{x\to 3} \frac{\frac{2}{x-2} - 2}{2(x-3)}$ still $\frac{0}{0}$

$= \displaystyle\lim_{x\to 3} \frac{\frac{-2}{(x-2)^2}}{2} = -1$

A. 0

B. $+\infty$

C. -1

D. $-\infty$

E. 1

Alternatively:

$\dfrac{\frac{2}{x-2} - 2}{2(x-3)} = \dfrac{2 - 2x + 4}{2(x-3)}$

$= \dfrac{6 - 2x}{2x - 6} = -1$ if $x \neq 3$

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5. *[4 marks]*

$\lim_{x \to +\infty} \left(1 + 2^{-x}\right)^{2^x} =$

A. 1

B. −1

C. e

D. $\frac{1}{e}$

E. $+\infty$

$$y = (1 + 2^{-x})^{2^x}$$

$$\ln y = 2^x \ln(1 + 2^{-x}) = \frac{\ln(1 + 2^{-x})}{2^{-x}} \quad \frac{0}{0}$$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\frac{1}{1+2^{-x}} \cdot \cancel{(2^{-x} \ln 2)(-1)}}{\cancel{2^{-x} \ln 2\,(-1)}}$$

$$= 1$$

$$\ln y \to 1$$

$$y = e^{\ln y} \to e^1 = e$$

6. *[4 marks]*

Every one knows that $e = 2.718\ldots$, but if you use Newton's method to estimate a solution of the equation $\ln x = 1$, starting with $x_0 = 1$, then (to two decimal places) $x_2 =$

A. 3

B. 2.83

C. 2.72

D. 2.61

E. 2

$$f(x) = \ln x - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\ln x_n - 1}{\frac{1}{x_n}}$$

$$x_0 = 1$$

$$x_1 = 1 - \frac{\ln 1 - 1}{1} = 2$$

$$x_2 = 2 - \frac{\ln 2 - 1}{\frac{1}{2}}$$

$$= 4 - 2\ln 2$$

$$\approx 2.6137\ldots$$

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7. *[4 marks]*

Let $F(x)$ be the function such that $F(0) = 0$, $F'(0) = 1$ and $F''(x) = e^x - x$. Then $F(x)$ is:

A. $e^x - x^3 - \frac{x^2}{2} - \frac{x}{6}$

B. $xe^x - x + 1$

C. $e^x - \frac{x^3}{6} - 1$

D. $e^x - xe^x$

E. $xe^x - x^2e^x$

$F'(x) = e^x - \frac{x^2}{2} + C$

$1 = F'(0) = e^0 + C$ so $C = 0$

$F'(x) = e^x - \frac{x^2}{2}$

$F(x) = e^x - \frac{x^3}{6} + K$

$0 = F(0) = e^0 + K$ so $K = -1$

8. *[4 marks]*

If $F(x) = \int_e^x \sqrt{\ln t}\, dt$, then $F'(e^2) =$

A. $\sqrt{2} - 1$

B. $\sqrt{2}$

C. $\frac{1}{4e^2} - \frac{1}{2e}$

D. $\frac{1}{4e^2}$

E. 0

$F'(x) = \sqrt{\ln x}$

$F'(e^2) = \sqrt{\ln e^2}$

$= \sqrt{2 \ln e}$

$= \sqrt{2}$

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9. *[4 marks]*

$$\int_0^1 \frac{e^{2x}}{4+e^{2x}}\,dx =$$

A. $\ln\left(\frac{4+e^2}{5}\right)$

B. $\ln\sqrt{4+e^2} - \frac{1}{2}\ln 5$

C. $\ln(4+e^2) - \ln 4$

D. $\frac{1}{2}\ln(4+e^2) - \ln 2$

E. $2\ln(4+e^2)$

Let $u = 4+e^{2x}$ $du = 2e^{2x}dx$

$$\int = \frac{1}{2}\int_5^{4+e^2} \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|\Big|_5^{4+e^2}$$

$$= \frac{1}{2}\ln(4+e^2) - \frac{1}{2}\ln 5$$

$$= \ln\sqrt{4+e^2} - \frac{1}{2}\ln 5$$

10. *[4 marks]*

$$\int_1^4 |x-2|\,dx =$$

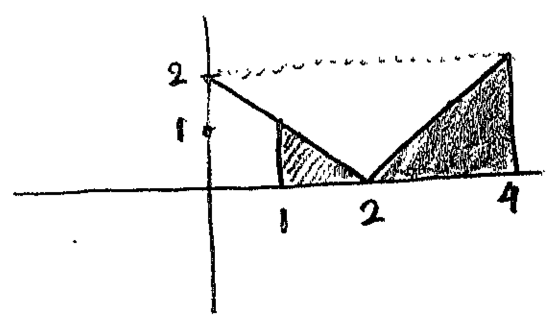
A. 0.5

B. 1.5

C. 2.5

D. −1.5

E. 3.5



$$\text{Area} = \frac{1}{2}\times 1\times 1 + \frac{1}{2}\times 2\times 2 = 2.5$$

or $$\int = \int_1^2 (2-x)dx + \int_2^4 (x-2)dx$$

$$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[(4-2)-(2-\tfrac{1}{2})\right] + (8-8) - (2-4)$$

$$= \frac{1}{2} + 2 = 2.5$$

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PART B. Written-Answer Questions

1. *[16 marks]*

$$\text{Given: } f(x) = \frac{x^3}{x^2-4},$$
$$\text{where } f'(x) = \frac{x^2(x^2-12)}{(x^2-4)^2}$$
$$\text{and } f''(x) = \frac{8x(x^2+12)}{(x^2-4)^3}.$$

[3] (a) Find all vertical and horizontal asymptotes (justify your answers).

$\boxed{\text{H.A. : none}}$, since deg numerator > deg denom so no finite limit at ∞ or $-\infty$.

$\boxed{\text{V.A. } x = 2 \text{ and } x = -2}$ since denom = 0 but num is not

[4] (b) Find where f is increasing, decreasing and all relative extrema.

	f	f'
$(-\infty, -2\sqrt{3})$	+	inc
$(-2\sqrt{3}, -2)$	−	dec
$(-2, 0)$	−	dec
$(0, 2)$	−	dec
$(2, 2\sqrt{3})$	−	dec
$(2\sqrt{3}, \infty)$	+	inc

$\wedge$ $x = -2\sqrt{3}$ local max

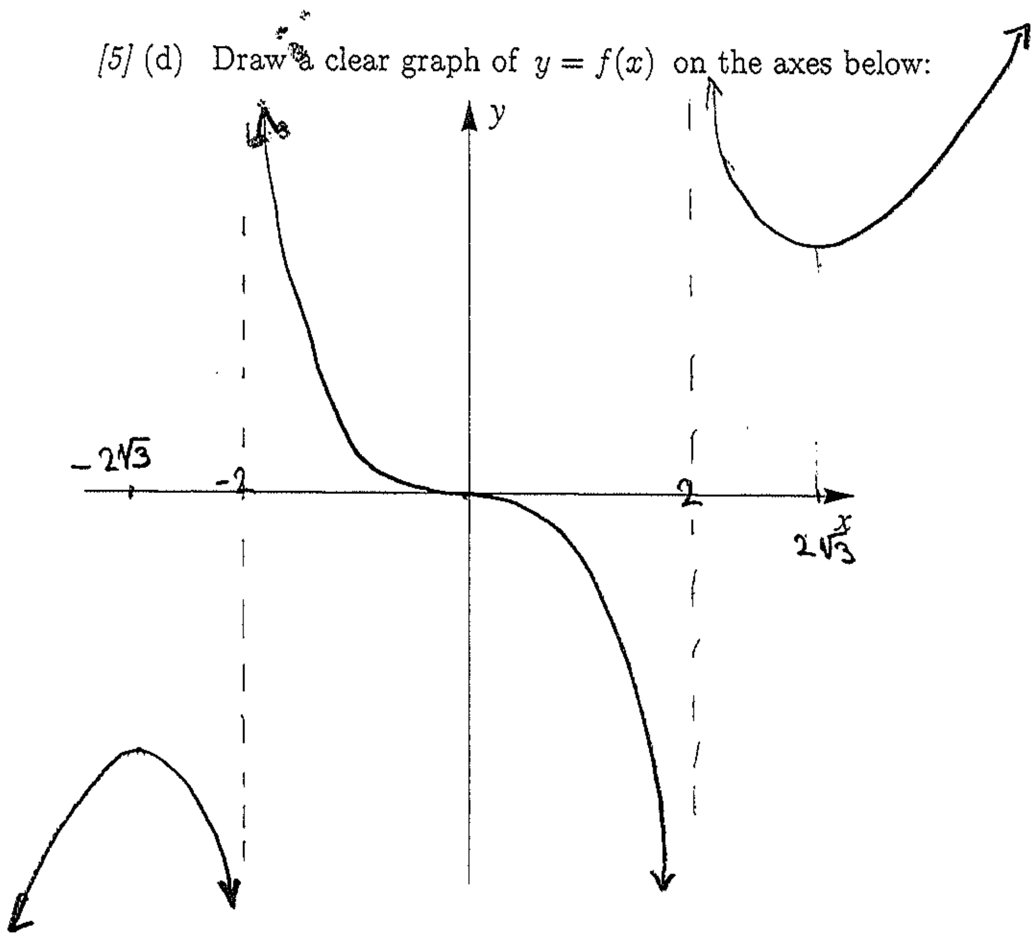
$\vee$ $x = 2\sqrt{3}$ local min

[4] (c) Find where f is concave upwards, concave downwards and all inflection points.

	f''	conc f
$(-\infty, -2)$	−	down
$(-2, 0)$	+	up
$(0, 2)$	−	down
$(2, \infty)$	+	up

$x = 0$ is the only point of inflection

[5] (d) Draw a clear graph of $y = f(x)$ on the axes below:



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2. *[15 marks]*

The price per unit of a product is given by the demand function $p = 1000e^{-q/100}$ with p in dollars.

[5] (a) What is the point elasticity of demand when $q = 25$?

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q\frac{dp}{dq}} = \frac{1000e^{-q/100}}{q(-10e^{-q/100})} = \frac{-100}{q}$$

$\boxed{\eta = -4}$ when $q = 25$

[10] (b) What price, to the nearest penny, maximizes revenue?

$$r = pq = 1000qe^{-q/100}$$

$$\frac{dr}{dq} = 1000e^{-q/100} - 10qe^{-q/100}$$

$$= 10e^{-q/100}(100-q)$$

$\frac{dr}{dq} = 0$ when $q = 100$

This maximizes q since $\frac{dr}{dq} < 0$ for all $q > 100$

and $\frac{dr}{dq} > 0$ for all $q < 100$.

r increases until $q = 100$ and decreases forever after.

At $q = 100$,

$p = 1000e^{-1} \approx \$\boxed{367.88}$

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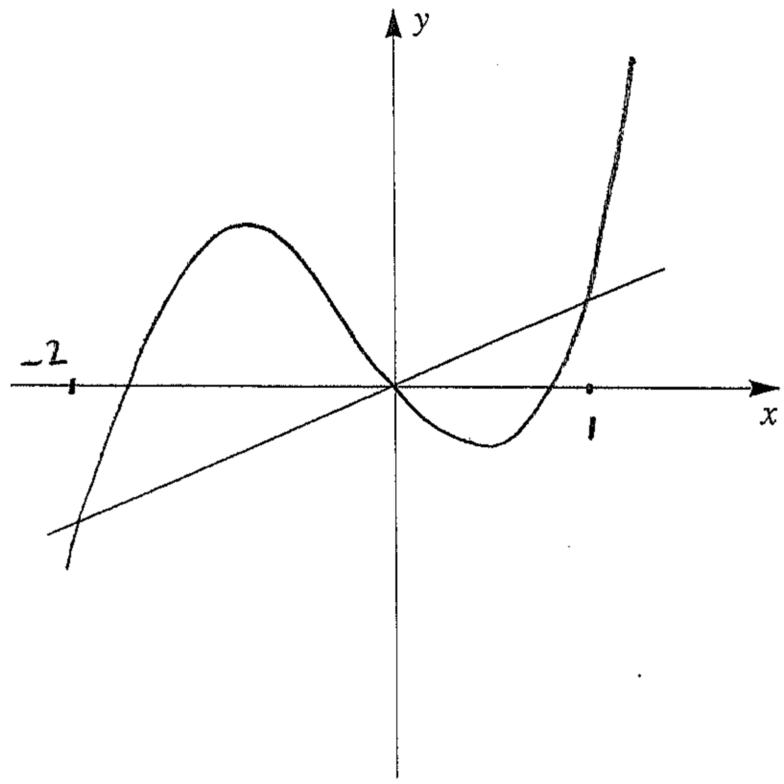
3. *[12 marks]*

Sketch a rough graph of the two curves $y = x^3 + x^2 - x$ and $y = x$ on the axes below and determine the area bounded by them.



The graph does not have to be this accurate. It is only necessary to know that $y = x^3 + x^2 - x$ lies above $y = x$ from $x = -2$ to 0 and below $y = x$ after that till $x = 1$.

points of intersection:

$$x^3 + x^2 - x = x$$
$$x^3 + x^2 - 2x = 0$$
$$x(x^2 + x - 2) = 0$$
$$x(x+2)(x-1) = 0 \qquad x = 0,\ x = -2,\ x = 1$$

$$\text{Area} = \int_{-2}^{0} (x^3 + x^2 - 2x)\,dx + \int_{0}^{1} (2x - x^2 - x^3)\,dx$$
$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_{0}^{1}$$
$$= -\left(\frac{16}{4} - \frac{8}{3} - 4\right) + \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$
$$= \frac{8}{3} + \frac{5}{12}$$
$$= \boxed{\frac{37}{12}}$$

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4. *[17 marks]*

Evaluate the integrals:

[8] (a) $\int x(\ln x)^2\,dx$

Let $u = (\ln x)^2$ $\quad dv = x\,dx$

$du = \frac{2\ln x}{x}\,dx$ $\quad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}(\ln x)^2 - \int x \ln x\,dx$$

Let $u = \ln x$ $\quad dv = x\,dx$

$du = \frac{dx}{x}$ $\quad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}(\ln x)^2 - \left[\frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx\right]$$

$$= \boxed{\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C}$$

[9] (b) $\int \frac{4x^2 - 6x - 6}{(x-1)^2(x+3)}\,dx$

$$= \int\left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}\right]dx = \int\left[\frac{1}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x+3}\right]dx$$

(see calculation below)

$A(x-1)(x+3) + B(x+3) + C(x-1)^2 = 4x^2 - 6x - 6$

$x = 1$: $4B = -8$ $\quad B = -2$

$x = -3$ $\quad 16C = 48$ $\quad C = 3$

$x = 0$, say $\quad -3A + 3B + C = -6$

$-3A - 6 + 3 = -6$

$-3A = -3$

$A = 1$

$$\int = \boxed{\ln|x-1| + 3\ln|x+3| + \frac{2}{x-1} + C}$$