Solved

Department of Mathematics
University of Toronto

TUESDAY, November 24, 2009 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1072	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	SS2127
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0201A	M3A	SS1086	T0701A	F2A	MP 118
T0201B	M3B	RW 142	T0701B	F2B	SS2105
T0201C	M3C	LM 157	T0701C	F2C	LM 155
T0201D	M3D	SS2110	T0701D	F2D	RW 143
T0301A	T3A	SS2105	T0801A	F3A	SS2111
T0301B	T3B	SS1074	T0801B	F3B	SS1088
T0301C	T3C	UC 163	T0801C	F3C	RW 143
T0401A	W9A	SS1072	T5101A	M5A	SS2110
T0401B	W9B	SS1088	T5101B	M5B	SS2128
T0501A	W3A	BA B024	T5101C	M5C	RW 229
T0501B	W3B	BA3004	T5101D	M5D	LM 123
T0501C	W3C	GB 404	T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

Which of the following matrices has an inverse?

A. $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 4 \end{pmatrix}$ not square

B. $\begin{pmatrix} 0 & -2 \\ 0 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so has an inverse (D)

E. none has an inverse

2. *[4 marks]*

For what value of a is the matrix $\begin{pmatrix} 2 & 3 \\ 1 & a \end{pmatrix}$ **not** invertible?

A. $a = 0$

B. $a = 1$

C. $a = \frac{3}{2}$

D. $a = 3$

E. no value of a

$R_1 \leftrightarrow R_2 \longrightarrow \begin{pmatrix} 1 & \alpha \\ 0 & 3-2\alpha \end{pmatrix}$

$R_2 \to R_2 - 2R_1$

Not invertible if $3 - 2\alpha = 0$

$\alpha = \frac{3}{2}$ (C)

NAME: _____ STUDENT NO: _____

3. *[4 marks]*

$\lim_{x \to 4^+} \frac{\sqrt{x^2-16}}{x+2} =$ $\frac{\sqrt{4^2-16}}{6} = 0$ (B)

A. $\frac{1}{6}$

B. 0

C. does not exist

D. $\frac{1}{4}$

E. 1

4. *[4 marks]*

$\lim_{x \to -\infty} \frac{1+2x-x^3}{x^3+2} = \frac{\lim x^3\left(\frac{1}{x^3}+\frac{2}{x^2}-1\right)}{x^3\left(1+\frac{2}{x^3}\right)} = -\frac{1}{1} = -1$ (B)

A. does not exist

B. -1

C. 1

D. $-\frac{1}{2}$

E. 2

NAME: _____ STUDENT NO: _____

5. *[4 marks]*

The solution to the inequality

$$\frac{x^2-5x+6}{\ln x} \leq 0$$

is given by:

A. $0 < x < 1$ or $2 \leq x \leq 3$

B. $1 < x \leq 2$ or $x \geq 3$

C. $x < 1$ or $2 \leq x \leq 3$

D. $x \leq 2$ or $x \geq 3$

E. $x < 0$ or $2 \leq x \leq 3$

$\frac{(x-3)(x-2)}{\ln x} \leq 0$ $x>0$ to start with so $\ln x$ defined.

fcn cont and not 0 on $(0,1), (1,2), (2,3), (3,\infty)$

point	$\frac{1}{2}$	$\frac{3}{2}$	e	10
sign	$-$	$+$	$-$	$+$

$(0,1)$ $[2,3]$

(A)

6. *[4 marks]*

$\lim_{x\to 4} \frac{x^{5/2}-32}{x-4} =$

(Hint: the limit is a derivative.)

A. 40

B. 25

C. 10

D. 20

E. 15

If $f(x) = x^{5/2}$

$f'(4) = \lim_{x\to 4} \frac{x^{5/2}-4^{5/2}}{x-4}$

$= \lim_{x\to 4} \frac{x^{5/2}-32}{x-4}$

But $f'(x) = \frac{5}{2}x^{3/2}$

so $f'(4) = \frac{5}{2}\cdot 4^{3/2} = 20$

(D)

NAME: _____ STUDENT NO: _____

7. *[4 marks]*

If $f(x) = \dfrac{x\sqrt{x^2+3}}{x+1}$, then $f'(1) =$

A. 1

B. $\dfrac{3}{4}$

C. $\dfrac{7}{4}$

D. $\dfrac{3}{2}$

E. $\dfrac{5}{4}$

$f(x) = x(x^2+3)^{\frac{1}{2}}(x+1)^{-1}$

$f'(x) = (x^2+3)^{\frac{1}{2}}(x+1)^{-1} + x \cdot \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x\,(x+1)^{-1}$

$+ x(x^2+3)^{\frac{1}{2}}(-1)(x+1)^{-2}$ by prod. rule

$f'(1) = \dfrac{4^{\frac{1}{2}}}{2} + \dfrac{1}{4^{\frac{1}{2}}} \cdot \dfrac{1}{2} - \dfrac{4^{\frac{1}{2}}}{2^2}$

$= 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$ (B)

Can also be done by quotient rule (or logarithmic diff. which we hadn't done yet.)

8. *[4 marks]*

If $f(x) = \dfrac{1}{1+\frac{1}{x+1}}$, then $f'(x) =$

A. $\dfrac{x}{(x+2)^2}$

B. $\dfrac{x}{x+2}$

C. $-\dfrac{1}{x+2}$

D. $-\dfrac{x}{(x+2)^2}$

E. $\dfrac{1}{(x+2)^2}$

$= -\dfrac{1}{\left(1+\frac{1}{x+1}\right)^2} \cdot \dfrac{-1}{(x+1)^2}$

$= \dfrac{1}{(x+1+1)^2} = \dfrac{1}{(x+2)^2}$ (E)

or $f(x) = \dfrac{x+1}{x+2} = 1 - \dfrac{1}{x+2}$

$f'(x) = \dfrac{1}{(x+2)^2}$ (E)

(or quotient rule)

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

Let $u(x)$ be a function such that $u(2) = 3$ and $u'(2) = 4$.
If $f(x) = x\,(u(x))^2$ then $f'(2) =$

A. 57
B. 33
C. 48
D. 84
E. 21

$f'(x) = [u(x)]^2 + 2x\,u(x)\,u'(x)$

$f'(2) = [u(2)]^2 + 2\cdot 2\,u(2)\,u'(2)$

$= 3^2 + 4\cdot 3\cdot 4$

$= 57$ (A)

10. *[4 marks]*

If a country's consumption function is given by

$$C = \frac{10\sqrt{I} + 0.8\sqrt{I^3} - 0.3I}{\sqrt{I}}$$

where I is income in billions of dollars, then the marginal propensity to save when $I = 25$ is

A. -0.83
B. 1.83
C. 0.5
D. 0.23
E. 0.77

$C = 10 + .8I - .3I^{\frac{1}{2}}$

$\frac{dC}{dI} = .8 - \frac{.3}{2} I^{-\frac{1}{2}}$

$\left.\frac{dC}{dI}\right|_{I=25} = .8 - \frac{.3}{2}\cdot\frac{1}{5} = .77$

$\frac{dS}{dI} = 1 - \frac{dC}{dI} = .23$ (D)

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[14 marks]*

[9] (a) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -3R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -6 & -2 & -3 & 1 & 0 \\ 0 & -4 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to -\frac{1}{4}R_2]{R_2 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -6 & -2 & -3 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \to 6R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & -2 & -\frac{3}{2} & 1 & -\frac{3}{2} \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right) \quad \begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} & -\frac{3}{4} \end{array}$$

$$\xrightarrow{R_1 \longleftrightarrow} \quad -R_3 - 2R_2 + R_1$$

The inverse is $\begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}$

Check by multiplication. ✓

[5] (b) Show that the solution to (a) above may be used to solve the system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 5 \\ 3x & & & + & z & = & 5 \\ x & - & 2y & + & z & = & 1 \end{array}$$

If $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ $\quad A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix}$

Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} + \frac{5}{2} - \frac{1}{4} \\ \frac{5}{4} + 0 - \frac{1}{4} \\ \frac{15}{4} - \frac{5}{2} + \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$x = 1, \ y = 1, \ z = 2$ which also checks in the equations.

NAME: _____ STUDENT NO: _____

2. *[18 marks]*

Given:

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x < 0 \\ x - 1 & \text{if } x = 0 \\ \dfrac{1 - 4^{1/x}}{1 + 4^{1/x}} & \text{if } 0 < x \leq 2 \\ \dfrac{\frac{1}{x-1} - 1}{x - 2} & \text{if } x > 2 \end{cases}$$

Find (show all steps):

[3] (a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{1 - 4^{\frac{1}{x}}}{1 + 4^{\frac{1}{x}}} = \lim_{x \to 0^+} \dfrac{4^{\frac{1}{x}}\left(4^{-\frac{1}{x}} - 1\right)}{4^{\frac{1}{x}}\left(4^{-\frac{1}{x}} + 1\right)}$

$x \to 0^+ \Rightarrow -\frac{1}{x} \to -\infty \Rightarrow 4^{-\frac{1}{x}} \to 0$ so get $-\frac{1}{1} = \boxed{-1}$

[3] (b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{|x|}{x} = \lim_{x \to 0^-} -\dfrac{x}{x} = \boxed{-1}$

[4] (c) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \dfrac{1 - (x-1)}{x - 2} = \dfrac{2 - x}{x - 2} = \boxed{-1}$

[4] (d) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \dfrac{1 - 4^{\frac{1}{x}}}{1 + 4^{\frac{1}{x}}} = \dfrac{1 - 4^{\frac{1}{2}}}{1 + 4^{\frac{1}{2}}} = \boxed{-\dfrac{1}{3}}$

[2] (e) whether f is continuous at $x = 0$ (justify your answer)

$f(0) = 0 - 1 = -1$ so from (a) and (b)

$\lim_{x \to 0^+} f(x) = f(0) = \lim_{x \to 0^-} f(x) = -1$ $\boxed{f \text{ is cont. at } x = 0}$

[2] (f) whether f is continuous at $x = 2$ (justify your answer)

from (c) + (d) $\lim_{x \to 2^+} f(x) = -1 \neq -\frac{1}{3} = \lim_{x \to 2^-} f(x)$

so $\boxed{f \text{ is } \underline{\text{not}} \text{ cont. at } x = 2}$

NAME: _____ STUDENT NO: _____

3. *[13 marks]*

Find an equation of the line tangent to the curve

$$y = \frac{(x^3+2)\sqrt{x+1}}{x^4+2x} = \frac{(x^3+2)\sqrt{x+1}}{x(x^3+2)}$$

at the point on the curve with $x = 1$.

$$= \frac{\sqrt{x+1}}{x}$$

(if you don't notice this, lots of diff. to do but answer the same)

$$y' = \frac{\frac{x}{2\sqrt{x+1}} - \sqrt{x+1}}{x^2}$$

$$= \frac{x - 2(\sqrt{x+1})^2}{2x^2\sqrt{x+1}} = \frac{-(x+2)}{2x^2\sqrt{x+1}}$$

$$y'(1) = -\frac{3}{2\sqrt{2}} \text{ or } -\frac{3}{4}\sqrt{2}$$

$$y(1) = \sqrt{2}$$

Eqn of line in point-slope form:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - \sqrt{2} = -\frac{3}{2\sqrt{2}}(x-1)} \text{ or } \boxed{y - \sqrt{2} = -\frac{3}{4}\sqrt{2}(x-1)}$$

or $2\sqrt{2}y - 4 = -3x + 3$

$$\boxed{2\sqrt{2}y + 3x = 7}$$

NAME: _____ STUDENT NO: _____

4. *[15 marks]*

A manufacturer has found that when m employees are working, the number of units of product produced per day is

$$q = 10\sqrt{m^2 + 4900} - 700$$

The demand equation for the product is

$$8q + p^2 - 19,300 = 0$$

where p is the selling price when the demand for the product is q units per day.

[8] (a) Determine the manufacturer's marginal-revenue product when $m = 240$.

$m = 240 \Rightarrow q = 10\sqrt{(240)^2+4900} - 700 = 10\sqrt{62,500} - 700$
$= 1800$

$p = \sqrt{19,300 - 8q}$ $\quad r = q\sqrt{19,300 - 8q} = 70 \times 1800$ when $m = 240$

$\frac{dr}{dq} = \sqrt{19,300-8q} - \frac{8q}{2\sqrt{19,300-8q}} = \frac{38,600 - 24q}{2\sqrt{19,300-8q}} = \frac{-4600}{140}$ at $m = 240$

$\frac{dr}{dq} = -\frac{230}{7}$ at $m = 240$ (

$\frac{dq}{dm} = \frac{10m}{\sqrt{m^2+4900}} = \frac{10m}{\sqrt{(240)^2+4900}} = \frac{2400}{250} = 9.6$ at $q = 240$

$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} = -\frac{230}{7} \times 9.6 \approx \boxed{-315.42857}$

[4] (b) Find the relative rate of change of revenue with respect to the number of employees when $m = 240$.

$\frac{1}{r}\frac{dr}{dm} = \frac{1}{70 \cdot 1800} \times \left(-\frac{230}{7}\right) \times 9.6$

$\approx \boxed{-.0025 \text{ or } -.25\%}$

[3] (c) Suppose it would cost the manufacturer \$400 more per day to hire an additional employee. Would you advise the manufacturer to hire the 241st employee? Why or why not?

No! If he hires an additional employee, he will decrease his revenue and increase his cost.