Solved

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Department of Mathematics
University of Toronto

Tuesday, November 30, 2010, 6:10 - 8:00 PM

MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1074	T0601A	R4A	SS2106
T0101B	M9B	SS1084	T0601B	R4B	LM 123
T0201A	M3A	LM 155	T0601C	R4C	SS2110
T0201B	M3B	RW 143	T0601D	R4D	LM 155
T0201C	M3C	SS2127	T0701A	F2A	LM 123
T0201D	M3D	SS1083	T0701B	F2B	LM 157
T0301A	T3A	SS1070	T0701C	F2C	AP 120
T0301B	T3B	SS1084	T0701D	F2D	RW 229
T0301C	T3C	WI524	T0801A	F3A	MP 134
T0401A	W9A	SS1074	T0801B	F3B	SS1074
T0401B	W9B	SS1084	T0801C	F3C	WI 524
T0401C	W9C	SS1088	T5101A	M5A	MP 134
T0501A	W3A	LM 123	T5101B	M5B	SS1073
T0501B	W3B	LM 157	T5101C	M5C	LM 155
			T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

$\lim_{x\to 1} \dfrac{x^2-x-2}{x^2-1}$

As $x \to 1$, $x^2-1 \to 0$; but $x^2-x-2 \to -2$.
There is no limit. (E)

A. $=-\dfrac{1}{2}$

B. $=\dfrac{1}{2}$

C. $=\dfrac{3}{2}$

D. $=-\dfrac{3}{2}$

E. does not exist

2. *[4 marks]*

$\lim_{x\to\infty} \dfrac{6x^3-1}{(2x^2+x+1)(x-3)} = \lim_{x\to\infty} \dfrac{6\left(1-\frac{1}{x^3}\right)}{\left(2+\frac{1}{x}+\frac{1}{x^2}\right)\left(1-\frac{3}{x}\right)} = \dfrac{6}{2\cdot 1} = 3$

(D)

A. $=-\infty$

B. $=0$

C. $=2$

D. $=3$

E. does not exist

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3. *[4 marks]*

If $f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 1 \text{ and } x \neq 0 \\ 2x - 1 & \text{if } 1 \leq x < 3 \\ 4 - x^2 & \text{if } x \geq 3 \end{cases}$

f has discontinuities at

A. $x = 1$ only

B. $x = 3$ only

C. $x = 0$ and $x = 1$ only

D. $x = 0$ and $x = 3$ only

E. $x = 0$ and $x = 1$ and $x = 3$ only

f is not defined at $x=0$, so **discont. at $x=0$**

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x-1 = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x} = 1$

f is cont at $x=1$.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 2x-1 = 5$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4-x^2 = -5$

f is **discont at $x=3$**

(D)

4. *[4 marks]*

Suppose f is continuous everywhere, and $f(-3) = 6$, $f(-2) = 0$, $f(0) = 4$, $f(1) = 0$, $f(2) = -3$. If $f(x) = 0$ **only** when $x = -2$ and $x = 1$, which of the following is false?

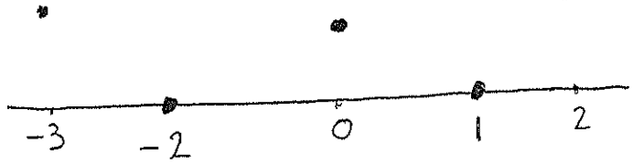
A. $f(-6) > 0$

B. $f(-1) > 0$

C. $f(3) < 0$

D. $f(4) > 0$

E. $f(-5) > 0$



The given info. says

	f
$(-\infty, -2)$	> 0
$(-2, 1)$	> 0
$(1, \infty)$	< 0

A, B, C, E are all true; but $f(4) < 0$ so (D) is false

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5. *[4 marks]*
If $\frac{\ln x}{x-6} \geq 0$ then

A. $x \leq 1$ or $x > 6$

B. $1 \leq x < 6$

C. $0 < x < 6$

D. $0 < x \leq 1$ or $x > 6$

E. $x \geq 1$

$\frac{\ln x}{x-6}$ is defined only when $x > 0$ and $x \neq 6$

$\frac{\ln x}{x-6} = 0$ at $x = 1$.

On $(0,1)$, $(1,6)$ and $(6,\infty)$ $\frac{\ln x}{x-6}$ is cont. and never zero, so checking one pt. in each interval is sufficient.

$(0,1)$: $\frac{\ln \frac{1}{2}}{\frac{1}{2}-6} = \frac{-}{-} = +$

$(1,6)$: $\frac{\ln e}{e-6} = \frac{1}{-} = -$

$(6,\infty)$: $\frac{\ln 10}{10-6} = \frac{+}{+} = +$

$(0,1] \cup (6,\infty)$

(D)

6. *[4 marks]*
If $f(x) = x^2(x+1)^5$, then $f'(1) =$

A. 128

B. 96

C. 120

D. 144

E. 216

$f'(x) = 2x(x+1)^5 + 5x^2(x+1)^4$

$f'(1) = 2 \cdot 2^5 + 5 \cdot 2^4$

$= 64 + 80 = 144$ (D)

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7. *[4 marks]*

$\frac{d}{dx}\left[\frac{x^3}{x^2+x+1}\right] =$ $\frac{(x^2+x+1)\cdot 3x^2 - x^3(2x+1)}{(x^2+x+1)^2}$

$= \frac{x^4+2x^3+3x^2}{(x^2+x+1)^2}$ (E)

A. $\frac{3x^4+2x^3+x^2}{(x^2+x+1)^2}$

B. $\frac{5x^4+4x^3+3x^2}{(x^2+x+1)^2}$

C. $\frac{3x^4+2x^3+x^2}{x^2+x+1}$

D. $\frac{5x^4+4x^3+3x^2}{x^2+x+1}$

E. $\frac{x^4+2x^3+3x^2}{(x^2+x+1)^2}$

8. *[4 marks]*

If $f(x) = \sqrt{8+e^{5x}}$, then $f'(0) =$

$f'(x) = \frac{1}{2\sqrt{8+e^{5x}}} \cdot 5e^{5x}$

$f'(0) = \frac{1}{2\sqrt{8+1}} \cdot 5e^0$

$= \frac{5}{6}$ (A)

A. $\frac{5}{6}$

B. $\frac{5}{3}$

C. $\frac{1}{6}$

D. $\frac{5}{2}$

E. $\frac{1}{3}$

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9. *[4 marks]*
If $f(x) = e^x \ln x$

$f'(x) =$ $e^x \ln x + \frac{e^x}{x} = \frac{e^x}{x}(x \ln x + 1)$ (A)

A. $\frac{e^x(1 + x \ln x)}{x}$

B. $\frac{(1 + x^2 e^{x-1} \ln x)}{x}$

C. $\frac{e^x(1 + \ln x)}{x}$

D. $\frac{e^x}{x}$

E. $\frac{1 + xe^x}{x}$

10. *[4 marks]*
If $f(x) = \frac{\ln x}{e^x + 1}$ then

$f'(x) = \frac{\frac{e^x + 1}{x} - e^x \ln x}{(e^x + 1)^2}$

$f'(1) =$

A. 0

B. $\frac{1}{e+1}$

C. 1

D. e

E. $\frac{1}{2}$

$f'(1) = \frac{e + 1 - e \cdot 0}{(e+1)^2}$

$= \frac{1}{e+1}$ (B)

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PART B. Written-Answer Questions

1. *[15 marks]*

Given the following function

$$f(x) = \begin{cases} \frac{|3+x|}{3+x} & \text{if } x < -3 \\ x + K & \text{if } -3 \le x \le 4 \ (K \text{ is a constant}) \\ \frac{\sqrt{x}-2}{x-4} & \text{if } x > 4 \end{cases}$$

Find (showing all steps in the solutions of a) to d) below)

[4] (a) $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{-(3+x)}{3+x} = \boxed{-1}$

[4] (b) The value of K in order for f to be continuous at $x = -3$.

$f(-3) = -3 + K = \lim_{x \to -3^+} -x + K = \lim_{x \to -3^-} f(x) = -1$ from (a)

$-3 + K = -1$

$\boxed{K = 2}$

[4] (c) $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{\sqrt{x}-2}{x-4} = \lim_{x \to 4^+} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{x-4} \cdot \frac{1}{\sqrt{x}+2}$

$= \boxed{\frac{1}{4}}$

Or: $\lim_{x \to 4^+} \frac{\sqrt{x}-2}{x-4} = f'(4)$ where $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$

[3] (d) The value of K in order for f to be continuous at $x = 4$.

$f(4) = 4 + K = \lim_{x \to 4^-} x + K = \lim_{x \to 4^+} f(x) = \frac{1}{4}$ from (c)

$4 + K = \frac{1}{4}$

$K = \boxed{-\frac{15}{4} = -3.75}$

(show all steps in solutions a)→d) above)

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2. *[15 marks]*

[8] (a) Find the equation of the line tangent to the curve $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ at the point where $x = 6$ and $y = 61$.

The equation of the line is

$y - 61 = m(x-6)$ where $m = \left.\frac{dy}{dx}\right|_{x=6}$

$\frac{dy}{dx} = 1 + x + \frac{x^2}{3}$ $\quad \left.\frac{dy}{dx}\right|_{x=6} = 1 + 6 + 18 = 25$

$y - 61 = 25(x-6)$

or $y = 25x - 89$

[7] (b) Find the point (x, y) on the graph of the curve $y = x \ln x$ at which the tangent line is horizontal.

$\frac{dy}{dx} = \ln x + \frac{x}{x} = \ln x + 1$

$\frac{dy}{dx} = 0$ when $\ln x = -1$

$x = e^{-1} = \frac{1}{e}$

$y = \frac{1}{e}\ln\left(\frac{1}{e}\right) = -\frac{1}{e}$

The point is $\left(\frac{1}{e}, -\frac{1}{e}\right)$

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3. *[15 marks]*

Let $w = u^{\frac{1}{3}}$ and $u = \dfrac{t+1}{t-1}$,

[8] (a) Find $\dfrac{dw}{dt}$ when $t = 2$ (to 2 decimal places)

When $t=2$, $u=3$.

$$\frac{dw}{dt} = \frac{dw}{du}\frac{du}{dt} = \frac{1}{3}u^{-\frac{2}{3}}\left[\frac{(t-1)-(t+1)}{(t-1)^2}\right]$$

$$= \frac{1}{3}u^{-\frac{2}{3}}\left(\frac{-2}{(t-1)^2}\right)$$

When $t=2$ and $u=3$

$$\frac{dw}{dt} = \boxed{\frac{-2}{3\cdot 3^{2/3}}} = -\frac{2}{3^{5/3}} \approx \boxed{-.32}$$

[3] (b) What is the relative rate of change of w with respect to t at $t = 2$?

$$\frac{1}{w}\frac{dw}{dt} = \frac{1}{u^{\frac{1}{3}}}\frac{dw}{dt} = \boxed{-\frac{2}{9}} \text{ at } t=2,\ u=3$$

$$\approx -.22$$

[4] (c) If $z = e^w$, find $\dfrac{dz}{dt}$ when $t = 2$ (to 2 decimal places)

$$\frac{dz}{dt} = \frac{dz}{dw}\frac{dw}{dt} = e^w\frac{dw}{dt}$$

At $t=2$, $w = 3^{\frac{1}{3}}$ and $\dfrac{dw}{dt} \approx -.32$

$$\frac{dz}{dt} = e^{3^{\frac{1}{3}}}(-.32)$$

$$\approx \boxed{-1.35 \text{ or } -1.36}$$

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4. *[15 marks]*

A company finds that when it sets its selling price at p dollars per kilogram, it will sell q kg per day, where $p^2(2q+1) = 10000$.

[5] (a) Find the company's (daily) demand and revenue functions ($p(q)$ and $r(q)$) in terms of q.

$p^2 = \frac{10,000}{2q+1}$ $\boxed{p = \frac{100}{\sqrt{2q+1}}}$

$r = pq$ $\boxed{r = \frac{100q}{\sqrt{2q+1}}}$

[5] (b) Find the company's (daily) marginal revenue when $q = 12$.

$$\frac{dr}{dq} = 100\left[\frac{\sqrt{2q+1} - \frac{q \cdot 2}{2\sqrt{2q+1}}}{2q+1}\right]$$

At $q=12$ $\frac{dr}{dq} = 100\left[\frac{5 - 24/10}{25}\right] = \boxed{10.4}$ or $\frac{52}{5}$

[5] (c) If the company hires $m \geq 1$ employees it can produce $q = 5m - 3$ kg per day. Find its (daily) marginal revenue product, $\frac{dr}{dm}$, when $m = 3$.

When $m=3$, $q=12$

$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm}$ and $\frac{dq}{dm} = 5$.

So, when $m=3$

$\frac{dr}{dm} = 5\frac{dr}{dq} = \boxed{52}$