Soln.

Department of Mathematics
University of Toronto

TUESDAY, October 28, 2008 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1074	T0501D	W3D	BA2139
T0101B	M9B	SS2105	T0601A	R4A	LM 157
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0101D	M9D	LM 158	T0701A	F2A	RW 229
T0201A	M3A	RW 229	T0701B	F2B	SS2111
T0201B	M3B	LM 157	T0701C	F2C	SS2128
T0201C	M3C	RW 142	T0801A	F3A	LM 155
T0201D	M3D	UC 52	T0801B	F3B	LM 123
T0301A	T3A	RW 143	T5101A	M5A	MP 134
T0301B	T3B	MP 134	T5101B	M5B	SS2111
T0401A	W9A	SS1074	T5101C	M5C	MP 118
T0401B	W9B	SS1086	T5101D	M5D	RW 143
T0401C	W9C	LM 158	T5201A	M6A	LM 161
T0501A	W3A	SS2105			
T0501B	W3B	SS2173			
T0501C	W3C	UC 256			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

If the rate of interest is 5% compounded continuously, then, in 10 years, \$100,000 will accumulate to

(A.) \$164,872

B. \$162,889

C. \$150,000

D. \$137,111

E. \$105,127

$100{,}000e^{.05\times 10} =$

$\$164.872$

2. *[4 marks]*

Hint: Note that 52 weeks = 1 year = 12 months.

The effective weekly rate of interest most nearly equivalent to an effective monthly rate of 1% is

A. 0.2182%

B. 0.2308%

C. 0.2351%

D. 0.2439%

(E.) 0.2299%

$(1.01)^{12} = (1+i)^{52}$

$i = (1.01)^{12/52} - 1$

$= .0022988686$

$\approx .002299$

NAME: _____ STUDENT NO: _____

3. *[4 marks]*

10 annual deposits of \$1,000 each are made into an account earning 5% compounded annually. 6 years after the last deposit, the account will amount to

A. \$13,206.79

B. \$16,855.58

C. \$13,400.96

D. \$12,577.89

E. \$16,288.95

$1000\, s_{\overline{10}|.05}\,(1.05)^6$

$= 1000\,\dfrac{[(1.05)^{10}-1]}{.05}\,(1.05)^6$

$= \$16,855.58$

4. *[4 marks]*

If a \$500,000 mortgage is amortized over 25 years at 6% compounded semi-annually with monthly payments, then each payment is closest to

A. \$3,199.03

B. \$5,042.08

C. \$15,002.11

D. \$28,713.94

E. \$39,113.36

$(1.03)^2 = (1+i)^{12}$

$500{,}000 = R\, a_{\overline{300}|i}$

$R = \dfrac{500{,}000\, i}{1-(1+i)^{-300}}$

$R = 500{,}000\,\dfrac{[(1.03)^{\frac{1}{6}}-1]}{1-(1.03)^{-50}}$

$R = \$3199.03$

NAME: _____ STUDENT NO: _____

5. *[4 marks]*

If a \$100 bond has 7 years until maturity, semi-annual coupons worth \$4 each, and an annual yield to maturity of 8%, then its market price is

A. \$67.02

B. \$79.17

C. \$92.54

D. \$93.95

(E.) \$100.00

$i = r$ so $P = V$

$P = \$100$

Alternatively:

$P = 100(1.04)^{-14} + 4a_{\overline{14}|.04}$

$= \$100$

6. *[4 marks]*

If a \$100 bond with semi-annual coupons has 9 years until maturity, an annual yield to maturity of 7%, and sells for \$90, then each semi-annual coupon is closest to

A. \$2.24

B. \$2.62

(C.) \$2.74

D. \$3.49

E. \$3.63

$90 = 100(1.035)^{-18} + (100r)\,a_{\overline{18}|.035}$

$100r = \dfrac{90 - 100(1.035)^{-18}}{a_{\overline{18}|.035}}$

$100r = 2.74$

NAME: _____ STUDENT NO: _____

7. *[4 marks]*

Let $X = \begin{pmatrix} 1 & -1 & 0 & 4 & 5 \\ 0 & 2 & 3 & 7 & 1 \end{pmatrix}$ $Y = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$.

Which one of the following is not defined?

A. Y^TX

B. Y^TYX

C. YXY

D. $(Y+Y^T)X$

E. YY^TYY^T

$_2Y^t{}_2X$ OK

$_2Y^t{}_2Y{}_2X$ OK

$_2Y{}_2X{}_2Y$

XY is undefined so C.

$_2(Y+Y^t)\,_2X$ OK

$_2Y{}_2Y^t{}_2Y{}_2Y^t$ OK

8. *[4 marks]*

When the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \\ 2 & -1 & 2 \end{pmatrix}$ is reduced, the number of non-zero rows is

A. 0

B. 1

C. 2

D. 3

E. 4

$R_2 \leftrightarrow R_1$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 2R_1$

$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$R_4 \leftrightarrow R_2 \longrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 3R_2$

3 non-zero rows

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

Consider the system

$$\begin{aligned} 2x &- y &+3z &-4w &+5v &= 0 \\ x &+2y &+2z &+ w &- v &= 0 \end{aligned}$$

The system has

A. only the trivial solution $x = y = z = w = v = 0$

B. a one-parameter family of solutions

C. a two-parameter family of solutions

D. a three-parameter family of solutions

E. a four-parameter family of solutions

Switching row 1 and row 2

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & -1 & 0 \\ 2 & -1 & 3 & -4 & 5 & 0 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & -1 & 0 \\ 0 & -5 & -1 & -6 & 7 & 0 \end{array}\right)$$

There will be 2 non-zero rows and 5 variables, hence $5 - 2 = 3$ parameters

10. *[4 marks]*

If $A^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

then the solution to the matrix equation $AX = C$ is given by

A. $x = 1$ $y = -3$ $z = -2$

B. $x = -1$ $y = -3$ $z = 3$

C. $x = 3$ $y = -3$ $z = 0$

D. $x = 0$ $y = -3$ $z = -1$

E. $x = 1$ $y = -3$ $z = 0$

$X = A^{-1}C$

$$X = \begin{pmatrix} -1 & 2 & 2 \\ 1 & -3 & -2 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1+4-2 \\ 1-6+2 \\ 1-2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

$x = 1$
$y = -3$
$z = 0$

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[15 marks]*

Ms. Clark wishes to accumulate \$10,000 by depositing \$100 each month into an account earning 3.6% per year compounded monthly.

[8] (a) How many \$100 deposits must she make before the account has more than \$9,900?

$9900 = 100\,s_{\overline{n}|.003}$

$99 = \dfrac{(1.003)^n - 1}{.003}$

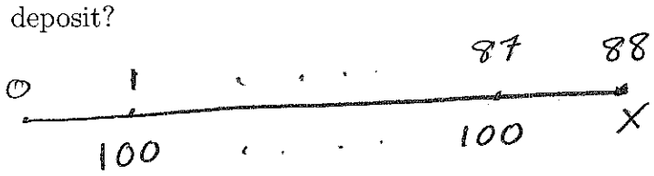
$.297 + 1 = (1.003)^n$

$1.297 = (1.003)^n$

$\ln(1.297) = n \ln(1.003)$

$n = \dfrac{\ln(1.297)}{\ln(1.003)} \approx 86.8$ so 86 not quite enough, but by 87 OK

[7] (b) Let n denote the answer to part (a). One month after her n^{th} \$100 deposit, Ms. Clark makes a last deposit to bring her account to exactly \$10,000. How much is this last deposit?



$= 10{,}000$

$100\,s_{\overline{87}|.003}(1.003) + X = 10{,}000$

$X = \$46.21$

NAME: _____ STUDENT NO: _____

2. *[15 marks]*

A person puts a down-payment of \$70,000 on a house and sets up a mortgage to pay for the rest. The mortgage is amortized over 20 years at 6.2% per annum compounded semi-annually, with monthly payments of \$915.00.

[7] (a) How much was the house worth at the start? $(1+i)^{12} = (1.031)^2$

$$A = 70{,}000 + 915\, a_{\overline{240}|i}$$

$$= 70{,}000 + 915\left[\frac{1-(1+i)^{-240}}{i}\right] = 915\left[\frac{1-(1.031)^{-40}}{(1.031)^{\frac{1}{6}}-1}\right] + 70{,}000$$

$$\boxed{A = \$196{,}477} \text{ to the nearest dollar}$$

[8] (b) How much more interest is contained in the 14th payment than in the 20th payment of the mortgage?

P.O. at beginning of 20th period $= 915\, a_{\overline{221}|i}$

Interest in 20th payment $= i \times 915\, a_{\overline{221}|i}$

Similarly interest in 14th payment $= i \times 915\, a_{\overline{227}|i}$

Interest in 14th − Interest in 20th

$$= 915\, i\left[a_{\overline{227}|i} - a_{\overline{221}|i}\right]$$

$$= 915\left[\left(1-(1+i)^{-227}\right) - \left(1-(1+i)^{-221}\right)\right]$$

$$= 915\left[(1+i)^{-221} - (1+i)^{-227}\right]$$

$$= 915\left[(1.031)^{-\frac{221}{6}} - (1.031)^{-227/6}\right]$$

$$= \boxed{\$8.94}$$

NAME: _____ STUDENT NO: _____

3. *[15 marks]*

Use the method of reduction of matrices to find all solutions of

$$\begin{array}{rcrcrcr} x & + & y & + & 4z & = & 1 \\ 2x & & -3y & + & 3z & = & -8 \\ 3x & & +2y & & +11z & = & 1 \\ 4x & & +2y & & +14z & = & 0 \end{array}$$

[No other method will be given any marks.]

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 2 & -3 & 3 & -8 \\ 3 & 2 & 11 & 1 \\ 4 & 2 & 14 & 0 \end{array}\right) \xrightarrow[R_4 \to R_4 - 4R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & -5 & -5 & -10 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & -2 & -4 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 \to R_3 + R_2 \\ R_4 \to R_4 + 2R_2}]{R_2 \to -\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{(optional)}} \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & y & z & \end{array}\right)$$

$x = -1 - 3z$

$y = 2 - z$

or

$x = -1 - 3r$

$y = 2 - r$

$z = r$

NAME: _____ STUDENT NO: _____

4. *[15 marks]*

An insurance company has three types of documents to process (types A, B and C). The following table shows how many hours each document needs with each of the accountant, lawyer and secretary.

	Accountant	Lawyer	Secretary
Document A	2	3	3
Document B	4	2	3
Document C	2	4	3

The accountant has a total of 34 hours, the lawyer has 35 hours, and the secretary has 36 hours to spend. How many documents of each type can they process?

Let A, B, C be the number of documents of type A, B and C respectively.

$$\left.\begin{aligned} 2A + 4B + 2C &= 34 \\ 3A + 2B + 4C &= 35 \\ 3A + 3B + 3C &= 36 \end{aligned}\right\}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 34 \\ 3 & 2 & 4 & 35 \\ 3 & 3 & 3 & 36 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_1 \to \frac{1}{2}R_1,\ R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & -4 & 1 & -16 \\ 0 & -3 & 0 & -15 \end{array}\right)$$

$$\xrightarrow[R_2 \to -\frac{1}{3}R_2]{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 1 & 0 & 5 \\ 0 & -4 & 1 & -16 \end{array}\right) \xrightarrow{R_3 \to R_3 + 4R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

So $C = 4$, $B = 5$ and $A = 17 - C - 2B = 17 - 4 - 10 = 3$

Type A = 3
Type B = 5
Type C = 4

Alternatively, proceeding to complete reduction

$$\xrightarrow{R_1 \to R_1 - R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

yielding the same answer