Soln

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Department of Mathematics
University of Toronto

Tuesday, October 26, 2010, 6:10 - 8:00 PM
MAT 133Y TERM TEST #1
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1074	T0601A	R4A	SS2106
T0101B	M9B	SS1084	T0601B	R4B	LM 123
T0201A	M3A	LM 155	T0601C	R4C	SS2110
T0201B	M3B	RW 143	T0601D	R4D	LM 155
T0201C	M3C	SS2127	T0701A	F2A	LM 123
T0201D	M3D	SS1083	T0701B	F2B	LM 157
T0301A	T3A	SS1070	T0701C	F2C	AP 120
T0301B	T3B	SS1084	T0701D	F2D	RW 229
T0301C	T3C	WI524	T0801A	F3A	MP 134
T0401A	W9A	SS1074	T0801B	F3B	SS1074
T0401B	W9B	SS1084	T0801C	F3C	WI 524
T0401C	W9C	SS1088	T5101A	M5A	MP 134
T0501A	W3A	LM 123	T5101B	M5B	SS1073
T0501B	W3B	LM 157	T5101C	M5C	LM 155
			T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

A professor gives \$100,000 to a university to provide for a yearly scholarship forever. If the money can be invested at 5% interest indefinitely, then the scholarship will be:

A. \$2,000,000/year

B. \$50,000/year

C. \$20,000/year

D. \$5,000/year

E. \$100,000/year

$R = rA$

$= .05 \times 100{,}000$

$= \boxed{\$5000/yr}$

2. *[4 marks]*

Which nominal annual interest rate compounded continuously is most nearly equivalent to (corresponds to the same effective annual rate as) 6% compounded semiannually?

A. 6.0%

B. 5.9%

C. 5.8%

D. 6.9%

E. 5.7%

$e^r = \left(1 + \frac{.06}{2}\right)^2 = 1.0609$

$r = \ln(1.0609) \approx .059$

$\boxed{5.9\%}$

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3. *[4 marks]*

A person makes equal annual deposits so that his account will have \$100,000 as soon as he makes the 8^{th} deposit. If the account earns 4% annually, the amount of each deposit (to the nearest \$100) is:

A. \$10,500

B. \$11,300

C. \$11,100

D. \$10,900

E. \$10,700

$100,000 = R\,s_{\overline{8}|.04}$

$R = \frac{100,000}{s_{\overline{8}|.04}} = \frac{100,000 \times .04}{(1.04)^8 - 1}$

$\approx 10,852.78$

$\boxed{\$10,900}$

4. *[4 marks]*

A couple can afford loan payments of \$2000 per month and they wish to pay off their loan in 15 years. If the interest rate on the loan is 4% compounded monthly then the size of the loan they can afford is closest to:

A. \$277,854

B. \$492,181

C. \$360,000

D. \$270,384

E. \$289,611

no. of payments $= 15 \times 12 = 180$

interest $= \frac{.04}{12}$

$A = 2000\, a_{\overline{180}|\frac{.04}{12}}$

$= 2000 \dfrac{\left[1 - \left(1 + \frac{.04}{12}\right)^{-180}\right]}{\frac{.04}{12}}$

$\approx 270,384.30$

$\boxed{\$270,384}$

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5. *[4 marks]*

If a \$6000 loan with annual interest 7% (compounded annually) is to be repaid with payments of \$X one year from now, \$2000 two years from now, and \$3X three years from now; then $X =$

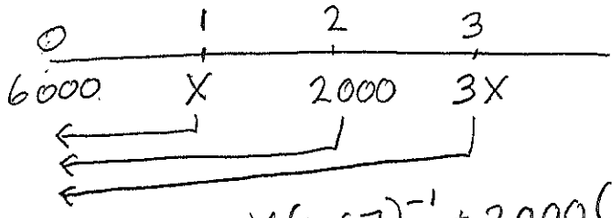
A. \$1302.88

B. \$1190.09

C. \$1211.35

D. \$1338.26

E. \$1257.03



$$6000 = X(1.07)^{-1} + 2000(1.07)^{-2} + 3X(1.07)^{-3}$$

$$X = \frac{6000 - 2000(1.07)^{-2}}{(1.07)^{-1} + 3(1.07)^{-3}}$$

$$X \approx 1257.03$$

6. *[4 marks]*

A bond with \$10,000 face value, an annual coupon rate of 7.6% with semi-annual coupons, an annual yield of 3.4%, and 6 years to maturity, will sell for (to the nearest \$):

A. \$12,262

B. \$10,389

C. \$12,245

D. \$10,000

E. \$9,437

$$P = 10{,}000(1.017)^{-12} + 380\, a_{\overline{12}|1.017}$$

$$\approx 12{,}262.30$$

\# 12,262

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7. *[4 marks]*

A 10-year \$200,000 mortgage with ***weekly*** payments of \$487.60 has an interest rate of 5% compounded semi-annually. How much principal is repaid in the 1st payment? (You may assume one year is exactly 52 weeks.)

A. \$295.29

B. \$190.03

C. \$384.61

D. \$297.57

E. \$279.27

$(1+i)^{52} = (1.025)^2 \qquad i = (1.025)^{\frac{1}{26}} - 1$

Interest in 1st payment $= 200{,}000\, i$

≈ 190.03

Principal $= R - 190.03$ where R satisfies

$200{,}000 = R\, a_{\overline{520}|i}$

$R = \dfrac{200{,}000\, i}{1-(1+i)^{-520}} = \dfrac{190.03}{1-(1.025)^{-20}}$

≈ 487.60

Principal in 1st payment $\approx 487.60 - 190.03$

$= \boxed{\$\,297.57}$

8. *[4 marks]*

If A is a 2×3 matrix, B is a 3×2 matrix, and C is a 3×4 matrix, then $(A^T + 2B)^T C$ is

A. a 2×3 matrix

B. a 3×2 matrix

C. a 2×4 matrix

D. a 3×4 matrix

E. not defined

$\underset{3}{\underset{2}{(}} A^T + 2B \underset{3}{\underset{2}{)}}$ is 3×2

$2\underset{2}{\underset{3}{(}}\ \underset{3}{\underset{2}{)}}^T \underset{3}{\underset{4}{C}} = 2\underset{2}{\underset{4}{(}}\ \underset{2}{\underset{4}{)}}$

$\boxed{2 \times 4}$

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9. *[4 marks]*

Which of the following is a solution to $AX = B$, where

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$
$$B = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}$$

A. $X = \frac{1}{2}\begin{pmatrix} 17 & -10 \\ 3 & 12 \end{pmatrix}$

B. $X = \begin{pmatrix} 13 & 10 \\ 6 & 3 \end{pmatrix}$

C. $X = \frac{1}{3}\begin{pmatrix} 20 & 10 \\ 16 & 8 \end{pmatrix}$

D. $X = \frac{1}{7}\begin{pmatrix} 15 & 3 \\ 3 & -6 \end{pmatrix}$

E. $X = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

$X = A^{-1}B$

$$\left(\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{array}\right)$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$X = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} = \boxed{\frac{1}{3}\begin{pmatrix} 20 & 10 \\ 16 & 8 \end{pmatrix}}$$

10. *[4 marks]*

The system

$$\begin{cases} x + 2y + 7z = 6 \\ 2y + 4z = 1 \\ x + 2y + (b+1)z = 8 \end{cases}$$

has no solution if b is:

A. 5

B. 6

C. 7

D. 8

E. 9

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & 6 \\ 0 & 2 & 4 & 1 \\ 1 & 2 & b+1 & 8 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 7 & 6 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & b-6 & 2 \end{array}\right)$$

No solution only if $\boxed{b=6}$

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PART B. Written-Answer Questions

1. *[15 marks]*

A $500,000 mortgage is amortized over 25 years with monthly payments.

[5] (a) If the interest rate is 4% compounded semi-annually then find the size of the monthly payment. $(1+i)^{12} = (1.02)^2$ $i = .00330589$

$$500{,}000 = R\, a_{\overline{300}|i}$$

$$R = \frac{500{,}000}{a_{\overline{300}|i}} = \frac{500{,}000\, i}{1-(1+i)^{-300}} = \frac{500{,}000\, i}{1-(1.02)^{-50}} \approx \boxed{\$2630.10}$$

[5] (b) Find the outstanding principal at the end of 5 years.

60 payments made. 240 remain.

$$P.O.I. = R\, a_{\overline{240}|i} = 2630.10\,\frac{[1-(1.02)^{-40}]}{[(1.02)^{\frac{1}{6}}-1]} \approx \boxed{\$435{,}269.41}$$

[5] (c) In 5 years the mortgage must be renewed. At that time the interest rate is 3% compounded monthly. If the remaining principal is amortized over the remaining 20 years, then find the new monthly payment.

Now $(1+i)^{12} = (1.015)^2$ $i = .00248456$

$$435{,}269.41 = R\, a_{\overline{240}|i}$$

$$R = \frac{435{,}269.41\, i}{1-(1+i)^{-240}}$$

$$= 435{,}269.41 \times \frac{[(1.015)^{\frac{1}{6}}-1]}{[1-(1.015)^{-40}]}$$

$$\approx \boxed{\$2409.95}$$

Because of the phrasing of the question, $i = \frac{.03}{12} = .0025$

and $R = \frac{435{,}269.41 \times .0025}{1-(1.0025)^{-240}} \approx \$2414$ was also accepted.

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2. *[15 marks]* (Long question)

[10] (a) Find the inverse of

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

(Don't be afraid if there are lots of fractions in your answer.)

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 3 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 6 & -9 & -3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2']{R_2 \to \frac{1}{6}R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 4 & 1 & -\frac{1}{3} & 1 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{4}R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 + \frac{3}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{24} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \end{array}\right) \xrightarrow[-3R_3]{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & \frac{7}{24} & -\frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{24} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \end{array}\right)$$

$\frac{3}{2}R_3 = \frac{3}{8} \quad -\frac{1}{8} \quad \frac{3}{8}$

$$A^{-1} = \begin{pmatrix} \frac{1}{8} & \frac{7}{24} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{24} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \end{pmatrix}$$

[5] (b) ***Using your answer in (a)***, solve the system

$$\begin{cases} x - y + 3z = 1 \\ 3x + 3y = 2 \\ 2y + z = -1 \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{7}{24} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{24} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{8} + \frac{14}{24} + \frac{3}{8} \\ -\frac{1}{8} + \frac{2}{24} - \frac{3}{8} \\ \frac{1}{4} - \frac{2}{12} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 26/24 \\ -10/24 \\ -\frac{2}{12} \end{pmatrix}$$

$x = \frac{13}{12}$

$y = -\frac{5}{12}$

$z = -\frac{1}{6}$

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3. *[15 marks]*

A fund manager is setting up a $150 million fund with stocks from hi-tech, energy, and media sectors. The manager wants twice as much (in dollars) hi-tech as media stock, and three times as much (in dollars) energy as hi-tech and media combined. How much (in dollars) should he/she buy from each sector?

Let H = amount of hi-tech
E = " " energy
M = " " media

$$\left.\begin{aligned} H+E+M &= 150 \text{ million} \\ H &= 2M \\ E &= 3(H+M) \end{aligned}\right\}$$

Easiest way: $H = 2M$
$E = 3(2M+M) = 9M$

$2M + 9M + M = 150,000,000$
$12M = 150,00,000$

M = $12,500,000
H = $25,000,000
E = $112,500,000

or

$$\begin{aligned} H+E+M &= 150 \\ H-2M &= 0 \\ 3H-E+9M &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} H & E & M & \\ 1 & 1 & 1 & 150 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 3 & 0 \end{array}\right)$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & -1 & -3 & -150 \\ 0 & -4 & 0 & -450 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & 1 & 3 & 150 \\ 0 & 0 & 12 & 150 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} H & E & M & \\ 1 & 0 & 0 & 25 \\ 0 & 1 & 0 & 112.5 \\ 0 & 0 & 1 & 12.5 \end{array}\right)$$

so H = 25 million
E = 112,5 million
M = 12.5 million

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4. *[15 marks]*

A debt of $50,000 is amortized at a nominal rate of 6% over 5 years with monthly payments, the first payment being at the end of the first month.

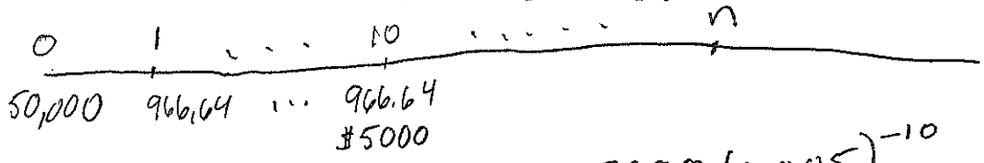
[3] (a) How much are the monthly payments?

$50{,}000 = R\, a_{\overline{60}|.005}$

$R = \dfrac{50{,}000 \times .005}{1-(1.005)^{-60}} \approx \boxed{\$966.64}$

Just after the tenth payment, the debtor is able to make an extra $5000 payment toward the debt. If the debtor continues to make regular payments, except for the last one which is smaller,

[6] (b) How many payments are required altogether to pay off the debt?



$50{,}000 = 966.64\, a_{\overline{n}|.005} + 5000(1.005)^{-10}$

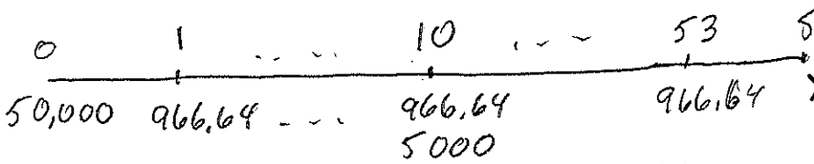
$\dfrac{1-(1.005)^{-n}}{.005} = \dfrac{50{,}000-5000(1.005)^{-10}}{966.64}$

$1-\dfrac{.005}{966.64}\left(50{,}000-5000\times 1.005^{-10}\right) = (1.005)^{-n}$

$n = -\dfrac{\ln(.76597668)}{\ln(1.005)} \approx 53.45$

Takes $\boxed{54 \text{ payments}}$ in all

[6] (c) How much is the last smaller payment?



Easiest way is to accumulate at 54:

$50{,}000(1.005)^{54} = 5000(1.005)^{44} + 966.64\, s_{\overline{53}|.005}(1.005) + X$

Subtracting: $\boxed{X \approx \$439.35}$

Note that $.45 \times 966.64 \approx \$435$. Even using unrounded $n = .45390\ldots$ gets $\$438.76$. Closer, but not quite right.