APM 236H1F term test 2

10 November, 2010

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: **No calculators or other aids allowed.**

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) Consider the following linear programming problem.
Minimize $z = x_1 + x_2 + x_3$ subject to the constraints

$$\begin{array}{ccccccc} 2x_1 & & & + & x_3 & = & 2 \\ 4x_1 & + & x_2 & + & 2x_3 & \geq & 7 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) (2 marks) Put the problem in **canonical form**.
(b) (7 marks) Find **all basic solutions** (feasible and infeasible) of the **canonical form** of the problem.
(c) (2 marks) Find **all extreme points** of the feasible region of the problem **given above.** Note that the above problem has **3** decision variables.
(d) (2 marks) **Solve** the problem **given above**.

(a) Maximize $Z = -x_1 - x_2 - x_3$ s.t.

$$\begin{array}{l} 2x_1 + x_3 = 2 \\ 4x_1 + x_2 + 2x_3 - x_4 = 7 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{array}$$

(b) The equality constraints have coefficient matrix
$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}$$
$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 4 & 1 & 2 & -1 \end{bmatrix}.$$

$\{A_1, A_3\}$ and $\{A_2, A_4\}$ are both linearly dependent sets, so the problem in (a) has only 4 basic solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \left(\begin{array}{c}\{x_1, x_2\} \\ \text{basic}\end{array}\right), \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix} \left(\begin{array}{c}\{x_1, x_4\} \\ \text{basic}\end{array}\right), \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} \left(\begin{array}{c}\{x_3, x_2\} \\ \text{basic}\end{array}\right), \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \end{bmatrix} \left(\begin{array}{c}\{x_3, x_4\} \\ \text{basic}\end{array}\right).$$

(c) Discarding the infeasible basic solutions (those having a negative component) and dropping x_4, the extreme points are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

(d) The respective objective values are 4 and 5. (using $z = x_1 + x_2 + x_3$)

$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is the optimal solution.

2. (13 marks) **Solve** the problem: Maximize $z = -x_1+2x_2+3x_3$ subject to the constraints

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & \le & 5 \\ -x_1 & + & 2x_2 & + & x_3 & \le & 2 \\ 2x_1 & + & x_2 & + & 2x_3 & \le & 12 \end{array}, \quad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

Tableau 1

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	2	-1	1	1	0	0	5
x_5	-1	2	1	0	1	0	2
x_6	2	1	2	0	0	1	12
	1	-2	-3	0	0	0	0

Tableau 2

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	3	-3	0	1	-1	0	3
x_3	-1	2	1	0	1	0	2
x_6	4	-3	0	0	-2	1	8
	-2	4	0	0	3	0	6

Tableau 3

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	-1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	1
x_3	0	1	1	$\frac{1}{3}$	$\frac{2}{3}$	0	3
x_6	0	1	0	$-\frac{4}{3}$	$-\frac{2}{3}$	1	4
	0	2	0	$\frac{2}{3}$	$\frac{7}{3}$	0	8

optimal tableau

3. (14 marks) **Solve the problem:** Maximize $z = 3x_1 + x_2 + 2x_3$ subject to the constraints

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & 2x_3 & = & 5 \\ x_1 & - & x_2 & - & x_3 & \leq & -1 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

phase 1, Tableau (1)

slack: x_4; artificial: y_1, y_2

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	1	1	2	0	1	0	5
y_2	-1	1	1	-1	0	1	1
	0	-2	-3	1	0	0	-6

phase 1, Tableau (2)

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	3	-1	0	2	1	-2	3
x_3	-1	1	1	-1	0	1	1
	-3	1	0	-2	0	3	-3

phase 1, Tableau (3)

	x_1	x_2	x_3	x_4	y_1	y_2	
x_1	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	1
x_3	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	2
	0	0	0	0	1	1	0

phase 2, Tableau (1)

	x_1	x_2	x_3	x_4	
x_1	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	1
x_3	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	2
	0	$-\frac{2}{3}$	0	$\frac{4}{3}$	7

phase 2, Tableau (2)

	x_1	x_2	x_3	x_4	
x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	2
x_2	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	3
	0	0	1	1	9

optimal tableau