Math 224
Linear algebra
December 2005

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1. Let $V = U \oplus W$ fom some subspaces U and W of a vector space V.

[4] (a) Show every vector v$\in V$ can be written uniquely as $v = u + w$ for some $u \in U$ and $w \in W$.

[4] (b) For v written uniquely as $v = u + w$ as in part (a) , define $T : V \to V$ by $T(v) = w$. Show that T is a linear transformation.

[4] (c) Show $U = \ker(T)$ and $W = \text{Im}(T)$.

[4] (d) Show $T^2 = T$.

SOLUTION:

(a) Let $v = u_1 + w_1 = u_2 + w_2$ where $u_1, u_2 \in U.w_1, w_2 \in W$.

Therefore $u_1 - u_2 = w_2 - w_1$, $u_1 - u_2 \in U, w_2 - w_1 \in W$. But we have a direct sum, thus the only vector in common is the zero vector. Therefore, $u_1 - u_2 = w_2 - w_1 = 0$.

Thus , $u_1 = u_2$ and $w_2 = w_1$. Therefore every vector can be written uniquely in this way.

(b) Let $v = u_2 + w_2, m = u_1 + w_1$ where $u_1, u_2 \in U.w_1, w_2 \in W$.

(i) $T(v+m) = T(u_1+w_1+u_2+w_2) = T(u_1+u_2+w_1+w_2) = w_1+w_2 = T(v) + T(m)$.

(ii) $T(kv) = T(ku_1 + kw_1) = kw_1 = kT(v)$

Thus, T is a linear transformation.

(c) $T(v) = T(u + w) = 0 \Rightarrow w = 0 \Rightarrow \ker(T) = U$

If $v \in W \Rightarrow T(v) = v \Rightarrow W \subseteq \text{Im}(T)$

Now , for any vector x, we have $x = u + w$ and $T(x) = w \in W$, thus $\text{Im}(T) \subseteq W$.

Therefore $\text{Im}(T) = W/$

(d) For any vector $x \in V$ we have $x = u + w$.

Now, $T^2(x) = T(T(x)) = T(w) = w$. So T and T^2 act the same on every vector in V. Thus, $T = T^2$.

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2. Let $\alpha = \{e^{3t}, te^{3t}, t^2e^{3t}, t^3e^{3t}\}$ be a basis of a vector space V of functions $f : R \to R$. Let $T : V \to V$ be defined by $T(f) = \frac{df}{dt}$.

   [8] (a) Find $[T]_{\alpha\alpha}$

   [8] (b) Let W be the T-cyclic subspace of V generated by $f(t) = te^t$. Find a basis for W and the characteristic polynomial of $T|_W$.

   SOLUTION:

   (a) $[T]_{\alpha\alpha} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, just think of derivatives of each element to get this.

   (b) $T(te^t) = e^t + te^t, T(e^t + te^t) = 2e^t + te^t$

   Thus $W = span\{te^t, e^t\}$

   Notice $T^2(te^t) = 2T(te^t) - te^t \Rightarrow (T^2 - 2T + I)(te^t) = 0$

   Thus characteristic polynomial is $f(x) = x^2 - 2x + 1$

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3. Let $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix}$ be the matrix of a linear transformation $T : R^3 \to R^3$ with respect to the standard basis of R^3. Compute $[T]_{\alpha\alpha}$ for the basis $\alpha = \{(1, 0, 1\}, (0, 1, 1), (1, 1, 0)\}$ of R^3.

   SOLUTION:

   Find change of basis matrix from α to standard.

   $[M] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$., Now, $[M]^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

   Thus, $[T]_{\alpha\alpha} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{5}{2} & \frac{3}{2} & 3 \\ \frac{3}{2} & \frac{5}{2} & 1 \end{pmatrix}$

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4. Suppose $\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ in the matrix of a linear transformation $T : R^3 \to R^3$ with respect to the standard basis of R^3.

[8] (a) Find an orthonormal basis α of R^3 consisting of eigenvectors of T.

[6] (b) Determine $[T]_{\alpha\alpha}$ and give the spectral decompostion of T.

SOLUTION:

(a) $\det\begin{pmatrix} -\lambda-1 & 2 & 2 \\ 2 & -\lambda-1 & 2 \\ 2 & 2 & -\lambda-1 \end{pmatrix} = -\lambda^3 - 3\lambda^2 + 9\lambda + 27 = 0$

$\Rightarrow -(\lambda-3)(\lambda+3)^2 = 0 \Rightarrow \lambda = \pm 3$

For $\lambda = 3 \Rightarrow \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is : $t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda = -3 \Rightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

Solution is : $t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, these two vectors are not orthogonal , so we must use gram-schmidt to orthogonalize them.

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, or $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

so we have orthogonal basis $\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$

Thus orthonormal basis is $\{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\} = \alpha$

(b) $[T]_{\alpha\alpha} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$,

The spectral decomposition is $\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \end{pmatrix}$

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5. Let $C[-\pi,\pi]$ be the vector space of continuous functions on $[-\pi,\pi]$.

[5] (a) Show that $< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product on $C[-\pi,\pi]$.

[5] (b) Show that the set $S = \{\frac{1}{\sqrt{2}}, \sin x, \cos x\}$ is an orthonormal set in $C[-\pi,\pi]$ with respect to the inner product in part (a).

[5] (c) Find the best least squares approximation to $f(x) = |x|$ on $[-\pi,\pi]$ by a trigonmetric function in span {S}.

SOLUTION:

(a) (i) $< f, f >= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx \geq 0$

(ii) $< f, f >= 0 \Leftrightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = 0 \Leftrightarrow f = 0$

(iii) $< f+g, h >= \frac{1}{\pi}\int_{-\pi}^{\pi}(f+g)hdx = \frac{1}{\pi}\int_{-\pi}^{\pi} fhdx + \frac{1}{\pi}\int_{-\pi}^{\pi} ghdx$

$=< f, h > + < g, h >$

(iv) $< kf, h >= \frac{1}{\pi}\int_{-\pi}^{\pi} kf(x)h(x)dx = k < f, h >$

(v) $< f, h >= \overline{< h, f >}$ (obviously true)

Thus, it is an inner product.

(b) $< \frac{1}{\sqrt{2}}, \sin x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{\sqrt{2}}dx = 0, < \frac{1}{\sqrt{2}}, \cos x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{2}}dx = 0$

$< \cos x, \sin x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \sin x dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\sin(2x)dx = 0$

Thus, it is an orthogonal set.

$< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}dx = 1, < \sin x, \sin x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(1-\cos(2x))dx = 1, < \cos x, \cos x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x dx = 1$

Thus, this is an orthonormal set.

(c) The approximation will be:

$< |x|, \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}} + < |x|, \sin x > \sin x + < |x|, \cos x > \cos x$

$= \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \frac{|x|}{\sqrt{2}}dx + \sin x(\frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin x dx) + \cos x(\frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos x dx)$

$= \frac{2}{\sqrt{2}\pi}\int_{0}^{\pi} \frac{x}{\sqrt{2}}dx + (\cos x)(\frac{2}{\pi}\int_{0}^{\pi} x\cos x dx) = \frac{1}{2}\pi - 4\frac{\cos x}{\pi}$

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6. Let $T : R^3 \to R^3$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$

[6] (a) Show that T is an isometry of R^3.

[6] (b) Determine if T is a rotation or reflection and find the axis of rotation or the fixed plane of reflection.

SOLUTION:

(a) $[T] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow [T]^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow [T][T]^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow TT^* = I$

Thus T is an isometry.

(b) $\det([T]) = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1 \Rightarrow$ indicates a rotation.

To find what you are rotating about look for eigenvector directions.

$\det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix} = 0 \Rightarrow \lambda^3 - 1 = (\lambda - 1)\left(\lambda^2 + \lambda + 1\right) = 0$

So , $\lambda = 1$ will give use direction.

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is : $t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Thus we are rotating around the line $x = y = z$.

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7. Let $T : V \to V$ be a normal linear transformation, that is $TT^* = T^*T$.

[4] (a) Shat that if $v \in \ker(T)$, then $v \in \ker(T^*)$.

[4] (b) Show that $T - \lambda I$ is a normal linear transformation for all $\lambda \in C$.

[4] (c) Show that if v is an eigenvector of T with eigenvalue λ, $T(v) = \lambda v$, then v is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

[4] (d) Show that if v and w are eigenvectors of T corresponding to distinct eigenvalue, then v and w are orthogonal.

SOLUTION:

(a) $v \in \ker(T) \Rightarrow T(v) = 0 \Rightarrow ||T(v)|| = 0 \Rightarrow < T(v), T(v) >= 0 \Rightarrow < v, T^*T(v) >= 0$

$\Rightarrow \overline{< TT^*(v), v >} = 0 \Rightarrow \overline{< T^*(v), T^*(v) >} = 0 \Rightarrow < T^*(v), T^*(v) >= 0$

$\Rightarrow ||T^*(v)|| = 0 \Rightarrow T^*(v) = 0 \Rightarrow v \in \ker(T^*)$.

(b) $(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \overline{\lambda} I) = TT^* - \overline{\lambda} T - \lambda T^* + \overline{\lambda}\lambda I$

$(T - \lambda I)^*(T - \lambda I) = (T^* - \overline{\lambda} I)(T - \lambda I) = T^*T - \overline{\lambda} T - \lambda T^* + \overline{\lambda}\lambda I = TT^* - \overline{\lambda} T - \lambda T^* + \overline{\lambda}\lambda I$

Thus they are the same, thus normal.

(c) Let $Tv = \lambda v$. First, $||Tv|| = \sqrt{< Tv, Tv >} = \sqrt{< v, T^*Tv >} = \sqrt{< v, TT^*v >}$

$= \sqrt{\overline{< TT^*v, v >}} = \sqrt{\overline{< T^*v, T^*v >}} = \sqrt{< T^*v, T^*v >} = ||T^*v||$ for any normal operator.

Now, since $T - \lambda I$ is normal we have $0 = ||(T - \lambda I)v|| = ||(T^* - \overline{\lambda} I)v||$. Thus v is an eigenvector of T^* and $\overline{\lambda}$ is an eigenvalue of T^*.

(d) Let $Tv = \lambda v, Tw = \alpha w \Rightarrow < Tv, w >=< \lambda v, w >= \lambda < v, w >$

But, $< Tv, w >=< v, T^*w >=< v, \overline{\alpha} w >= \alpha < v, w >$

Now, this implies $\alpha < v, w >= \lambda < v, w > \Rightarrow (\alpha - \lambda) < v, w >= 0 \Rightarrow < v, w >= 0$.

Thus v and w are orthogonal.