Math 224
Linear algebra
December 2006

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1. (10 marks) Let $W_1 = \{A \in M_{22}(R) | A = A^T\}$. Let $W_2 = \{(x, y, z, w) \in R^4 | x + y - z = 0\}$. Show W_1 and W_2 are isomorphic and find (define) an isomorphism $T : W_1 \to W_2$.

   SOLUTION:

   If $A = A^T \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow b = c.$

   Thus W_1 is the set of all matrices of the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$.

   Now, W_2 consists of the points of the form $(x, y, x + y, w)$.

   There are many choices for the isomorphism, so here is one.

   $f(\begin{pmatrix} a & b \\ b & d \end{pmatrix}) = (a, d, a + d, b)$. To prove this is a isomorphism we need to show this is linear, 1-1 and onto.

   a) linear:

   i) $f(k\begin{pmatrix} a & b \\ b & d \end{pmatrix}) = f(\begin{pmatrix} ka & kb \\ kb & kd \end{pmatrix}) = (ka, kd, ka{+}kd, kb) = kf(\begin{pmatrix} a & b \\ b & d \end{pmatrix})$

   ii) $f(\begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} e & f \\ f & g \end{pmatrix}) = f(\begin{pmatrix} a+e & b+f \\ b+f & d+g \end{pmatrix}) = (a+e, d+g, a+d+e+g, b+f)$

   $= (a, d, a + d, b) + (e, g, e + g, f) = f(\begin{pmatrix} a & b \\ b & d \end{pmatrix}) + f(\begin{pmatrix} e & f \\ f & g \end{pmatrix})$

   so linear

   b) onto:

   Pick $v \in W_2 \Rightarrow v = (x, y, x + y, w)$.

   Thus $\begin{pmatrix} x & w \\ w & y \end{pmatrix} \in W_1$ and $f(\begin{pmatrix} x & w \\ w & y \end{pmatrix}) = v$. Thus f is onto.

   c) 1-1:

Just have to show kernel is zero vector in W_1.

$f(\begin{pmatrix} a & b \\ b & d \end{pmatrix}) = (0,0,0,0) \Rightarrow (a, d, a+d, b) = (0,0,0,0) \Rightarrow a = b = c = d = 0$

$\Rightarrow \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ kernel is zero vector.

Therefore f is an isomorphism and thus the two spaces are isomorphic.

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2. (11 marks) Let V be an inner product space and $T : V \to V$ a linear operator. Prove that if T is normal, then T and T^* have the same image and the same kernel.

   SOLUTION:

   i) proof that kernels are the same. (Recall that normal means $T^*T = TT^*$)

   Let $v \in \ker(T) \Leftrightarrow Tv = 0 \Leftrightarrow < Tv, Tv >= 0$. But $< Tv, Tv >=< v, T^*Tv >=< v, TT^*v >= \overline{< TT^*v, v >} = \overline{< T^*v, T^*v >} =< T^*v, T^*v >$

   Thus $< T^*v, T^*v >= 0 \Leftrightarrow T^*v = 0 \Leftrightarrow v \in \ker(T^*)$. Thus T and T^* have the same kernels.

   ii) proof that images are the same.

   (assuming that V is finite dimensional) We can use dimension theorem.

   $\dim V = \dim($ Kernel$) + \dim($ Image$)$

   so we have $\dim V = \dim($ Kernel T$) + \dim($ Image T$) = \dim($ Kernel T$^*) + \dim($ Image $T^*)$

   Thus we have $\dim($ Image $T) = \dim($ Image $T^*)$

   We can form a basis for V consisting of basis of kernel and extend this to basis of V.

   Since they both have same kernel the extension is a basis for Image of both. Thus images are the same.

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3. (10 marks) Let α and β be the standard bases of $P_3(R)$ and $P_2(R)$ respectively. Suppose the linear transformation $T : P_3(R) \to P_2(R)$ is given by

$$[T]_{\beta\alpha} = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{pmatrix}$$

Find a basis β' of $P_2(R)$ such that the matrix $[T]_{\beta'\alpha}$ is the reduced row echlon form of $[T]_{\beta\alpha}$.

SOLUTION:

Determine the matrix which puts $[T]_{\beta\alpha}$ into row reduced echelon form. Think of the elementary operations that perform this operation.

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{pmatrix}, \text{ row echelon form: } \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

So this matrix takes us from the β basis to the β' basis. The inverse of this is more useful as we can read the β' basis directly from it.

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \text{ inverse: } \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

So $\beta' = \{(1, 2, -1), (1, 1, 0), (0, 0, 1)\}$
or $\beta' = \{1 + 2x - x^2, 1 + x, x^2\}$.

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4. Consider the inner product space $C[0, 1]$, the set of all continuous functions on [0,1], with inner product defined by $< f(x), g(x) >= \int_0^1 f(x)g(x)dx$.

[2] (a) Show that 1 and 2x-1 are orthogonal.

[4] (b) Determine ||1|| and ||2x-1||.

[6] (c) Find a function from the subspace $W = span\{1, 2x - 1\}$ that best approximates the function $f(x) = \sqrt{x}$.

SOLUTION:

(a) $< 1, 2x-1 >= \int_0^1 (2x-1)dx = (x^2-x)|_0^1 = 0$

Therefore they are orthogonal.

(b) $||1||^2 =< 1, 1 >= \int_0^1 dx = x|_0^1 = 1 \Rightarrow ||1|| = 1$

$||2x-1||^2 =< 2x-1, 2x-1 >= \int_0^1 (2x-1)^2 dx = \frac{1}{3} \Rightarrow ||2x-1|| = \frac{1}{\sqrt{3}}$

(c) The best approximation would be

$\frac{<\sqrt{x},1>}{1} + \frac{<\sqrt{x},2x-1>}{\frac{1}{3}}(2x-1) = \int_0^1 \sqrt{x}dx + 3(\int_0^1 (\sqrt{x})(2x-1)dx)(2x-1)$

$= \frac{2}{3} + \frac{2}{5}(2x-1) = \frac{4}{15} + \frac{4}{5}x$

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5. Consider C^2 with the standard inner product. Let $T : C^2 \to C^2$ be given by

$T(x, y) = (x + (1-i)y, (1+i)x + 2y)$.

[2] (a) Find $T^*(x, y)$

[8] (b) Find an orthonormal basis of C^2 consisting of eigenvectors of T.

SOLUTION:

a) $[T] = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} \Rightarrow [T]^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$

Thus $T = T^*$

b) $\det(\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = -3\lambda + \lambda^2 = 0$

Thus we have eigenvalues 0 and 3.

So for $\lambda = 0$ we must solve $\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Solution is : $\begin{pmatrix} -t_1 + it_1 \\ t_1 \end{pmatrix} = t_1 \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$

For $\lambda = 3$ we must solve $\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Solution is : $\begin{pmatrix} t_1 - it_1 \\ 2t_1 \end{pmatrix} = t_1 \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$

So we have basis $\{\begin{pmatrix} -1+i \\ 1 \end{pmatrix}, \begin{pmatrix} 1-i \\ 2 \end{pmatrix}\}$ which is orthogonal but they are not necessarily of length one.

$||\begin{pmatrix} -1+i \\ 1 \end{pmatrix}||^2 = (-1+i)(-1-i) + (1)(1) = 3$

$||\begin{pmatrix} 1-i \\ 2 \end{pmatrix}||^2 = (1-i)(1+i) + (2)(2) = 6$

Thus our orthonormal basis is $\{\frac{1}{\sqrt{3}}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1-i \\ 2 \end{pmatrix}\}$

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6. Let W_1 and W_2 be subspaces of a vector space V. Suppose $V = W_1 \oplus W_2$ and let $v = w_1 + w_2$ denote the decompostion of each $v \in V$ with $w_1 \in W_1$ and $w_2 \in W_2$. For $i = 1, 2$ let $P_i : V \to V$ be defined by $P_i(v) = w_i$.

   Prove the following statements.

   [3] (a) $P_i^2 = P_i$ for $i = 1, 2$.

   [3] (b) P_1P_2 is the zero transformation.

   [3] (c) $P_1 + P_2$ is the identity transformation.

   [3] (d) Each eigenvalue of $P_i, i = 1, 2$ is either 0 or 1.

   SOLUTION:

   (a) Consider $P_1^2 v = P_1^2(w_1 + w_2) = P_1(P_1(w_1 + w_2)) = P_1(w_1) = w_1$

   This is true for all $v \in V \Rightarrow P_1^2 = P_1$. Similary $P_2^2 = P_2$.

   (b) $P_1P_2(v) = P_1P_2(w_1 + w_2) = P_1(w_2) = P_1(0 + w_2) = 0 \Rightarrow P_1P_2$ is the zero transformation.

   (c) $(P_1 + P_2)(v) = P_1(v) + P_2(v) = P_1(w_1 + w_2) + P_2(w_1 + w_2) = w_1 + w_2 = v$

   $\Rightarrow P_1 + P_2$ is the identity transformation.

   (d) Consider $P_1(v) = \lambda v \Rightarrow P_1(w_1 + w_2) = \lambda(w_1 + w_2) \Rightarrow w_1 = \lambda(w_1 + w_2)$

   $\Rightarrow w_1(1 - \lambda) = \lambda w_2$. Now since the only common vector W_1 and W_2 have in common is the zero vector we must have $w_1(1-\lambda) = 0, \lambda w_2 = 0$.

Thus we have solutions $\lambda = 1$ with $w_2 = 0$ or $\lambda = 0$ with $w_1 = 0$.

Similary with P_2 we will only get these two eigenvalues again.

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7. Let $T : R^4 \to R^4$ be a linear transformation whose matrix relative to the standard basis of R^4 is given by

$$A = \begin{pmatrix} 4 & -1 & -19 & 3 \\ -3 & 0 & 12 & -2 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[7] (a) Find a basis for the T-cyclic subspace W generated by $X = (1, 1, 0, -1)$.

[3] (b) Find the characteristic polynomial $T|_W$.

SOLUTION:

(a) $$\begin{pmatrix} 4 & -1 & -19 & 3 \\ -3 & 0 & 12 & -2 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 & -19 & 3 \\ -3 & 0 & 12 & -2 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 & -19 & 3 \\ -3 & 0 & 12 & -2 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 & -19 & 3 \\ -3 & 0 & 12 & -2 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus a basis for the cyclic subspace is $\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$

$$\begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}\}$$

(b)

Now, these four vectors span the original space !

Thus no linear combination of these will give the zero vector (except the trivial case).

But, $T^4 v = 0$. This implies the minimal polynomial is $m(x) = x^4$.