University of Toronto
Complex Variables
Mat 334H1S
3 hours

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1. (12 marks) Sketch and describe as accurately as possible the image of the annulus $A = \{z | r_1 \leq |z| \leq r_2\}, 0 < r_1 < 1 < r_2$, under the mapping $z \to \log z$ where the log function is defined using the principal branch $-\pi < \arg z \leq \pi$

   SOLUTION:

   First consider $r_1 \leq |z| \Rightarrow z = r_1 e^{i\theta} \Rightarrow \log z = \ln r_1 + i\theta, -\pi < \theta \leq \pi$. Since $r_1 < 1 \Rightarrow \ln(r_1) < 0$. Thus this inner circle maps to the straight line with $x = \ln r_1$ and $y = \theta$ with $-\pi < \theta \leq \pi$.

   Thus, this maps the annulus to the square with $\ln(r_1) \leq x \leq \ln(r_2)$ and $-\pi < y \leq \pi$.

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2. (12 marks) Show $f(z) = \frac{z}{1+|z|}$ is not holomorphic anywhere.

   SOLUTION:

   Use Cauchy-Riemann conditions here.

   $f(z) = \frac{z}{1+|z|} = \frac{x+iy}{1+\sqrt{x^2+y^2}} = \frac{x}{1+\sqrt{x^2+y^2}} + i\frac{y}{1+\sqrt{x^2+y^2}} = u + iv$

   $u_y = \frac{d}{dy}(\frac{x}{1+\sqrt{x^2+y^2}}) = -\frac{xy}{\left(1+\sqrt{(x^2+y^2)}\right)^2\sqrt{(x^2+y^2)}}$

   $v_x = \frac{\partial}{\partial x}(\frac{y}{1+\sqrt{x^2+y^2}}) = -\frac{xy}{\left(1+\sqrt{(x^2+y^2)}\right)^2\sqrt{(x^2+y^2)}}$

   C-R$\Rightarrow u_y = -v_x \Rightarrow xy = 0 \Rightarrow x = 0$ or $y = 0$.

   But, these are straight lines. To be analytic at a point you need the function to satisfy the cauchy riemann equations in a neighbourhood of that point. There are no points which are analytic in a neighbourhood, thus this is not holomorphic anywhere.

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3. (12 marks) Classify all singularities, including at ∞, for the following functions

a) $f(z) = \frac{z^5}{z^3+z}$

b) $g(z) = \frac{e^z-1}{z(z-1)}$

SOLUTION:

a) $f(z) = \frac{z^5}{z(z^2+1)} \Rightarrow$ singularities at $z = 0, \mp i$

Now, as $\frac{z^5}{z(z^2+1)} = \frac{z^4}{(z^2+1)}, z \neq 0$. We can redefine $f(0) = 0$, thus $z = 0$ is a removable singularity.

$\lim\limits_{z \to i} (z-i)\frac{z^5}{z(z^2+1)} = \lim\limits_{z \to i} \frac{z^5}{z(z+i)} = \frac{1}{2i} \Rightarrow z = i$ is a simple pole.

$\lim\limits_{z \to -i} (z+i)\frac{z^5}{z(z^2+1)} = \lim\limits_{z \to -i} \frac{z^5}{z(z-i)} = \frac{-1}{2i} \Rightarrow z = -i$ is a simple pole.

For ∞ consider the function $f(\frac{1}{z}) = \frac{1}{z^5\left(\frac{1}{z^3}+\frac{1}{z}\right)} = \frac{1}{z^2(z^2+1)}$. Now we just have to consdier what happens for $z = 0$. Notice $\lim\limits_{z \to 0} z^2\frac{1}{z^2(z^2+1)} = 1$. Thus ∞ is a second order pole.

b) Notice $\lim\limits_{z \to 1} (z-1)\frac{e^z-1}{z(z-1)} = e - 1 \neq 0$. Thus $z = 1$ is a simple pole.

$z = 0$ a little tricker. $e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + ...$

Thus, $\frac{e^z-1}{z} = 1 + \frac{1}{2}z + \frac{1}{6}z^2 + ...$

Thus, $\lim\limits_{z \to 0} \frac{e^z-1}{z(z-1)} = -1$, thus $z = 0$ is a removable singularity.

For ∞ consider the limit $\lim\limits_{z \to \infty} \frac{1}{z^n}g(z), n = 0, 1, 2, 3, ...$ for any value of n this limit goes to ∞. Thus $z = \infty$ is an essential singularity.

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4. (12 marks) Expand the function $f(z) = e^{\frac{z}{z-2}}$ in a laurent series about $z = 2$ and determine the region of convergence of this series. What is the residue at $z = 2$.

SOLUTION:

$f(z) = e^{\frac{z}{z-2}} = f(z) = e^{\frac{z-2+2}{z-2}} = f(z) = e^{1+\frac{2}{z-2}} = ee^{\frac{2}{z-2}} = e(1 + \frac{2}{z-2} + (\frac{2}{z-2})^2\frac{1}{2!} + ...)$

$= e\sum_{n=1}^{\infty}(\frac{2}{z-2})^n\frac{1}{n!}$

Use ratio test to find radius of convergence.

$\lim_{z\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{z\to\infty} |(\frac{2}{z-2})^{n+1}\frac{1}{(n+1)!} \div (\frac{2}{z-2})^n \frac{1}{n!}| = \lim_{n\to\infty} \frac{2}{n+1}\frac{1}{z-2} = 0, z \neq 2$

Thus region of convergence is $z \neq 2$. (note...little trick here...look at the point where function is not defined. The radius of convergence will always be the distance from where you expand the it to the next singularity. Since there are none (except the point of expansion), the radius is infinity)

The residue at $z = 2$ is $2e$.

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5. (12 marks) Evaluate $\int_0^{2\pi} \frac{1}{2+\cos t} dt$

SOLUTION:

Let $z = e^{it} \Rightarrow \cos t = \frac{e^{it}+e^{-it}}{2} = \frac{z+\frac{1}{z}}{2}, dz = ie^{it}dt = izdt$

Thus $\int_0^{2\pi} \frac{1}{2+\cos t}dt = \oint_{|z|=1} \frac{1}{2+\frac{z+\frac{1}{z}}{2}} \frac{1}{iz}dz = -2\oint_{|z|=1} \frac{i}{4z+z^2+1}dz$

Now, $4z + z^2 + 1 = 0$, Solution is : $\left\{z = -2+\sqrt{3}\right\}, \left\{z = -2-\sqrt{3}\right\}$

Only $z = -2+\sqrt{3}$ is inside the unit circle , thus we need residue at this point.

$-2\oint_{|z|=1} \frac{i}{4z+z^2+1}dz = 2\pi i[\lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\frac{-2i}{4z+z^2+1}] =$

$2\pi i[\lim_{z\to -2+\sqrt{3}} \frac{-2i}{z+2+\sqrt{3}}] = \frac{4\pi}{2\sqrt{3}} = \frac{2}{3}\sqrt{3}\pi$

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6. (12 marks) Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^6+1}dx$

SOLUTION:

Integrate this by forming a semi-circular path in the upper half plane. This will give you $\oint_{\text{upper half plane}} \frac{1}{z^6+1}dz = \int_{-\infty}^{\infty} \frac{1}{x^6+1}dx$

$z^6 + 1 = 0 \Rightarrow z^6 = -1 = e^{\pi i + 2n\pi i} \Rightarrow z = e^{\frac{\pi i+2n\pi i}{6}}$

Therefore the poles in the upper half plane are $e^{\frac{\pi}{6}i}, e^{\frac{\pi}{2}i}, e^{\frac{5\pi}{6}i}$ (these are simple poles).

Residue for these at $z = a$ will equal $\frac{1}{6a^5}$ (this only works for simple poles).

Thus $\int_{-\infty}^{\infty} \frac{1}{x^6+1}dx = 2\pi i(\frac{1}{6e^{\frac{5\pi}{6}i}} + \frac{1}{6i^5} + \frac{1}{6e^{\frac{25\pi}{6}i}}) = \frac{8}{3}\frac{\pi}{4} = \frac{2}{3}\pi$

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7. (14 marks) Evaluate using residues P.V. $\int_{-\infty}^{\infty} \frac{\cos x}{4-x^2} dx$

SOLUTION:

We cannot simply replace 'x' with 'z' as the new function will blow up in the upper half plane. So consider $\int_{-\infty}^{\infty} \frac{e^{ix}}{4-x^2} dx$

$\oint_{\text{upper half plane}} \frac{e^{iz}}{4-z^2} dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{4-x^2} dx - \oint_{\text{around } z=2} \frac{e^{iz}}{4-z^2} dz - \oint_{\text{around } z=-2} \frac{e^{iz}}{4-z^2} dz$ (note: we have negative around $z = \pm 2$ since path goes clockwise around them)

$\oint_{\text{upper half plane}} \frac{e^{iz}}{4-z^2} dz = 0$ as no singularities inside the path.

$\oint_{\text{around } z=2} \frac{e^{iz}}{4-z^2} dz = (\frac{1}{2})(2\pi i) \lim_{z\to 2} (z-2) \frac{e^{iz}}{4-z^2} = \pi i (\frac{-e^{2i}}{4})$

$\oint_{\text{around } z=-2} \frac{e^{iz}}{4-z^2} dz = (\frac{1}{2})(2\pi i) \lim_{z\to -2} (z+2) \frac{e^{iz}}{4-z^2} = \pi i (\frac{e^{-2i}}{4})$

Thus, $0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{4-x^2} dx - \pi i (\frac{-e^{2i}}{4}) - \pi i (\frac{e^{-2i}}{4}) \Rightarrow$

$\int_{-\infty}^{\infty} \frac{e^{ix}}{4-x^2} dx = \pi i (\frac{-e^{2i}}{4}) + \pi i (\frac{e^{-2i}}{4}) = \frac{1}{2}\pi \sin 2$ (pure real)

Thus, $\int_{-\infty}^{\infty} \frac{\cos x}{4-x^2} dx = \frac{1}{2}\pi \sin 2$

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8. (14 marks) Evaluate using residues: $\int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx$

SOLUTION:

Use a semi-circular path in the upper half plane.

$\oint_{\text{upper half plane}} \frac{\log z}{(1+z^2)^2} dz = \int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx + \int_{-\infty}^{0} \frac{\log x}{(1+x^2)^2} dx +$ (integral that goes around 0 - but this goes to 0 anyway).

$\oint_{\text{upper half plane}} \frac{\log z}{(1+z^2)^2} dz = 2\pi i(\text{residue at } z = i) = 2\pi i \lim_{z\to i} \frac{d}{dz}(\frac{\log z}{(z+i)^2})$

$= 2\pi i \lim_{z\to i} \frac{\frac{(z+i)^2}{z} - 2(z+i)\log z}{(z+i)^4} = 2\pi i \lim_{z\to i} \frac{\frac{(z+i)}{z} - 2\log z}{(z+i)^3} = 2\pi i \frac{2-2(i\frac{\pi}{2})}{-8i} = -\frac{1}{2}\pi + \frac{1}{4} i\pi^2$

For $\int_{-\infty}^{0} \frac{\log x}{(1+x^2)^2} dx$ let $x = re^{i\pi}$ thus $\int_{-\infty}^{0} \frac{\log x}{(1+x^2)^2} dx = \int_{\infty}^{0} \frac{\log re^{i\pi}}{(1+r^2)^2} e^{i\pi} dr$

$= \int_0^{\infty} \frac{\log r + i\pi}{(1+r^2)^2} dr = \int_0^{\infty} \frac{\log r}{(1+r^2)^2} dr + i \int_0^{\infty} \frac{\pi}{(1+r^2)^2} dr$

Thus we have $-\frac{1}{2}\pi + \frac{1}{4} i\pi^2 = \int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx + \int_0^{\infty} \frac{\log r}{(1+r^2)^2} dr + i \int_0^{\infty} \frac{\pi}{(1+r^2)^2} dr$

Which gives us $\int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx = -\frac{1}{4}\pi$

And $\int_0^{\infty} \frac{\pi}{(1+r^2)^2} dr = \frac{1}{4}\pi^2$