Economic Theory manuscript No.
(will be inserted by the editor)

N-Organization Overlapping Generations Games with Memoryless Agents*

Kirk Bailey, Rodney Beard

School of Economics, University of Queensland, Brisbane, AUSTRALIA
e-mail: r.beard@economics.uq.edu.au

Received: date ; Revised version: date

Summary. This paper attempts to explain ongoing cooperation among infinitely-lived organizations controlled by finitely-lived agents. As such the folk theorem in overlapping generations (OLG) games with memoryless altruistic agents and communication is generalized to N-players and imperfect communication. For organizations characterized as sequences of agents possessing altruism towards the following agent, results for two cases of severe informational restraints are examined. In the first, players cannot observe play before entry to the game but are sent a publicly communicated message from their predecessor. In the second, that message with some probability is false. For both cases perfect public equilibrium(PPE) results obtain. We find that under perfect communication payoffs are time invariant, while under imperfect communication they vary but can approximate any feasible payoffs to an arbitrary degree.

Keywords and Phrases: Folk Theorem, Organizations, Overlapping Generations Games, Altruism, Memory, Communication, Repeated Games

JEL Classification Numbers: C72, D23, D43, D92

* We would like to thank those seminar participants whose comments have improved this paper.

Correspondence to: R. Beard

1 Introduction

There are many economic examples of long running organizations managed by a succession of agents. Firms, nations and unions are all organizations that can all outlive the tenure of a single administrator. Our intuition suggests that there must be some way to ensure long run cooperation among those organizations in spite of agents short term goals. This is exactly the question that folk theorems in overlapping generations(OLG) games address.

To do so a model of a repeated stage game played by overlapping generations of players is used. The model is similar to that of the folk theorem in repeated games [3] in that a set of players repeatedly interact in a stationary strategic environment defined by a triple of the set of players, payoff functions and strategies in the usual way. However in OLG games it is organizations that repeatedly interact in each period of the repeated game, and the decision making agent of each organization changes over time. So each player in the stage game only represents his organization for a finite number of periods.

One early model of OLG's with short lived agents acting in the long run interest can be found in Samuelson's 1958 paper[8]. He found that pareto optimal equilibria in his model were not necessarily stable given the best response of each generation of agents. Indeed he stated

> "The Golden Rule or Kant's Categorical Imperative (enjoining like people to follow the common pattern that makes each best off) are often not self enforcing: if all but one obey, the one may gain selfish advantage by disobeying - which is where the sheriff comes in: we politically invoke force on ourselves, attempting to make an unstable equilibrium a stable one."

This began an ongoing interest in modelling behaviour among OLG's strategically because in doing so the precise requirements of the stability of equilibria in Samuelson's and similar models could be examined.

The first to formalize Samuelson's problem was Hammond[4] who's model used strategies that obtain a sub-game perfect result. Cremer[2] broadened the setting away from Samuelson's model by using a generalized prisoners' dilemma in the stage game to obtain more general results. He also argued that OLG games provide a natural theory of long-lived organizations.

Kandori[5], Smith[10], and Salant[9] each presented variants of the folk theorem in OLG games. Their results utilized a fully generalized stage game which in the case of Kandori and Smith held for any finite number N of players in each generation. A notable recent result is Breton et al.[1], who study an overlapping generations game of team production in which the question of the revelation of individual worker productivity is addressed.

Unfortunately the generalized folk theorem's equilibrium strategy requires that perfect information of the entire game be available to all players. In an OLG setting this is demanding a lot of players upon their entry to the game. Consequently Lagunoff and Matsui[6] restrict players to only what they themselves have observed in the game. That is, players enter the game with no memory of previous play in the stage game. This restriction causes the generalised folk theorem to fail. However by giving the memoryless agents communication and altruism the result can be reversed. They found that where agents are altruistic to those that replace them in the stage game and where agents send messages to their successors, a folk theorem obtained. This folk theorem has the natural interpretation of conditions that ensure long run cooperation among organizations of short lived agents.

As outlined in their own paper, Lagunoff and Matsui's folk theorem is limited because it obtains for stage games of only two players. That is for OLG games among only two organizations.[1] They state

> "...we are confident a modification of the model to finite numbers of organizations can be done. What makes the modification nontrivial is that it is not only the length of the lifetimes, but also the overlap between any two individuals' lifetimes that matter."

In this paper the folk theorem is extended to N-organizations, broadening the applicability of previous results. The result obtains by reformulating the equilibrium strategies to specifically cater for heterogeneous players and by redefining the nature of the cost of sending each message. With respect to Lagunoff and Matsui's concern regarding the overlap between players lifetimes, the model also carefully specifies the phases of each agents life, and follows Kandori[5] in doing so.

It also seems natural to explore the nature of communication in this model and in doing so further restrict the information available to each player. As such, the model is extended to imperfect communication. This is the case where players observe with some positive probability a message other than the one intended. We find that in this case a folk theorem obtains with payoffs arbitrarily close to any equilibrium payoff profile of the perfect communication case.

Both folk theorems rely on public perfect equilibrium (PPE), although payoffs in the perfect communication case are strongly stationary in that they are the same each period, whereas payoffs in the imperfect communication case can vary.

This paper is constructed as follows. Section 2 outlines the OLG game. Section 3 presents the folk theorem for N organizations and its proof, which largely consists of the construction of equilibrium strategies. A section constructing a short numerical example then follows. Section 5 further generalizes the model to imperfect communication and presents a folk theorem, followed by its proof. Section 6 constructs a short numerical example for this case. Section 7 summarizes the paper's findings and concludes.

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[1] Also their folk theorem only applies to games with homogeneous agents. These are games where all players either receive payoffs above or below their equilibrium payoffs when they sanction other players.

2 The Model

2.1 The Stage Game

The repeated stage game played between the organizations each period is defined here in a general way. That is, no particular game such as the prisoner's dilemma is used. This broadens the scope of the models applicability to *any* suitable strategic setting. This stage game is the most basic element of the OLG game defined later.

The stage game is defined as $G = \langle N, (A_i), (u_i) \rangle$. Under this definition N denotes the finite set of players where $i \in N$ is some *player*. Each i has a non-empty set A_i of *actions* and a *payoff function* $u_i : A \to \Re$ where $A = \times_{j \in N} A_j$. The set $\Delta(A_i)$ denotes the set of probability distributions over (A_i) with a member of this set referred to as a *mixed strategy*.[2] The probability that $\alpha_i \in \Delta(A_i)$ assigns to a *pure* strategy $a_i \in A_i$ is denoted by $\alpha_i(a_i)$ and the support of α_i are those elements for which $\alpha_i(a_i) > 0$. Finally some profile $(\alpha_j)_{j \in N}$ will induce a probability distribution over the set A. Note that $\alpha = (\alpha_j)_{j \in N}$. Assuming players independently randomize, an *evaluation* by i is then defined as $\Sigma_{a \in A}(\prod_{j \in N} \alpha_j(a_j))u_i(a)$. This is also known as i's *expected payoff* and will be represented by u_i in the same manner as the payoff and will be simply referred to as the *payoff* from here on, no confusion should arise from this.

The *payoff profile* is defined as $u = (u_1, u_2, ..., u_N)$ where $u \in \Re^N$. A player i's *minimax* payoff denoted $\underline{u_i}$ is defined as the highest payoff he can guarantee himself regardless of the other $N-1$ players' choices. Further $\underline{u_i} = u_i(\underline{\alpha}^i)$ where

$$\begin{aligned} u_i(\underline{\alpha}^i) &= u_i(\underline{\alpha}^i_i, \underline{\alpha}^i_{-i}) \\ \underline{\alpha}^i_{-i} &\in \arg\min_{\alpha_{-i}} \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}) \\ \underline{\alpha}^i_i &\in \arg\max_{\alpha_i} u_i(\alpha_i, \underline{\alpha}^i_{-i}) \end{aligned}$$

Under the definition above then, the minimax payoff $\underline{u_i}$ is the minimum payoff level any $i \in N$ can be held down to. Of course for some player $j \in N$ where $j \neq i$ holding i down to his minimax level will give him a related payoff as well. This is known as the payoff from *minimaxing* or *sanctioning* i and is denoted $u_j(\underline{\alpha}^i)$.

The set of all payoff profiles in the stage game G is known as the *feasible* payoff set. We denote by V the set of feasible payoffs for which each player i receives strictly greater than his minimax payoff. Note $V \in \Re^N$. The stage game G is repeatedly played every period of the OLG game below. The nature of the N players that participate in each period is also formally given below.

[2] This will be abbreviated to simply *strategy* from here on.

2.2 The Overlapping Generations

In an OLG game the players of the stage game change over time. In effect each organization plays the infinitely repeated game, but in the stage game are only represented by an agent for a finite number of periods. After that time the player's successor in his organization replaces him. How and when players replace each other is outlined below.

In the OLG game time is discrete and is denoted $t = 1, 2, 3, ...$ In each period from the first onwards G is played by N players. These N players are finitely lived representatives drawn from N infinitely lived organizations. So the OLG game is played between N sequences of players. Each player from each organization plays the stage game for T periods after entering the game in some period t. They are then replaced by a successor after period $t + T$ who possesses the same payoff function and action set in the stage game. So while players enter and exit the game, the stage game itself remains the same. We indicate by $i(k)$ the k^{th} agent of the i^{th} organization.

We refer to the *normal* phase as those periods where all players are from the same *generation* and no player is going to exit soon. This is further defined below, but for example the normal phase consists of periods where $1(4)$, the 4^{th} agent of organization 1, plays $2(4)$, the 4^{th} agent of organization 2, and so on. As the k^{th} agent from each organization interacts with the k^{th} agents of the other organizations during the normal phase, we use k to represent a particular *generation* of agents.

The complete set of agents is denoted $\{1(k), 2(k), ..., N(k)\}$, where $k = 1, 2, ...$ In what follows note that when no confusion will arise an agent $i(k)$ will often be denoted simply by their organization i for convenience.

The games overlapping nature is forced by letting the players of the first generation live longer than T periods. This should not cause confusion and will not be referred to again. The number of periods from the exit of player $i-1(k)$ to the exit of player $i(k)$ is given by a vector $\mathbf{K} = \{K_1, K_2, ..., K_N\}$.

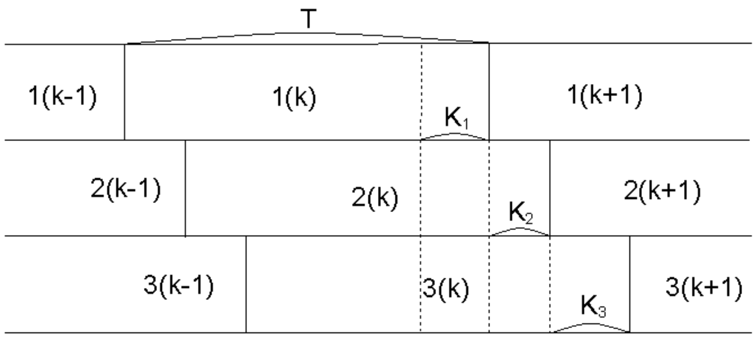

Fig. 1 An OLG Game with 3 Organizations

The interpretation of K_1 is that K_1 refers to the number of periods from the start of the *overlap* phase to the exit of player $i(k)$. This overlap phase is of length $K = K_1 + K_2 + ... + K_N$ and is a phase in the OLG game where players of generation k are slowly replaced by those of the generation $k+1$. Now refining the definition above, the other phase known as the *normal* phase lasts for $T - K$ periods. During the normal phase all players from generation k play the stage game, and $1(k)$ has at least K_1 periods of play left before he is replaced by his successor.

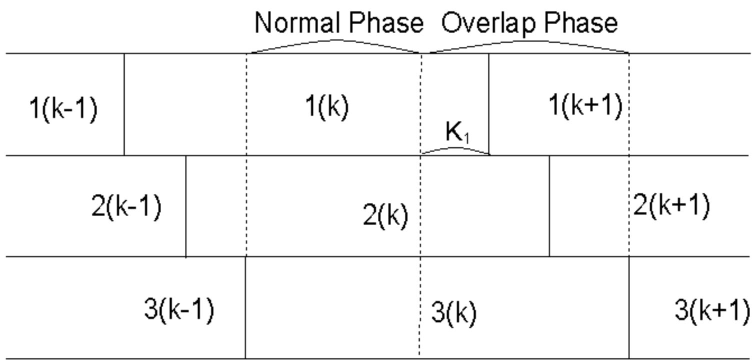

Fig. 2 The Phases in an OLG Game

2.3 The Overlapping Generations Game

There are three reasons why an OLG game is played among *organizations.* Firstly, players feel altruistic towards their successors, but only *their* successors. Secondly, players communicate to their memoryless successors by sending public messages, in fact they communicate to all the players alongside their successor as well. Finally, players share the same 'type' as their successors, that is the same payoff function and action set in the stage game. Only the third of these considerations has been mentioned as yet, how the model incorporates the first two will now be shown.

Players are *altruistic* as by assumption $i(k)$ takes as his payoff in the OLG game the sum of his T payoffs while playing the stage game G and the discounted sum of his successor's stage game payoffs. The discount factor $\delta \in (0, 1)$ measures the degree of altruism.

Incoming players are also defined as being *memoryless.* As such they depend on their predecessor's message to condition their strategy. Formally an agent $i(k)$ entering the OLG game in some period e has a personal history of the strategies played in the stage games up to period t given by $h^t_{i(k)} = (\alpha^e, \alpha^{e+1}, ...\alpha^{t-2}, \alpha^{t-1})$, where $t > e$. In period e itself $i(k)$'s personal history is the null history, h^0. The full set of personal histories at t is then given by $H^t_{i(k)}$ where $h^t_{i(k)} \in H^t_{i(k)}$.

Signals sent by outgoing agents enter agents' histories. The first of these comes from an agents predecessor $i(k-1)$ and is received in $i(k)$'s first period of play, some period e. For the remainder of that overlap phase $i(k)$ will continue to receive messages from outgoing agents. This is due to the *public* nature of communication in these games. Player $i(k)$ will receive his next message in K_{i+1} periods, when player $i+1(k-1)$ leaves the game. Another message will be received after K_{i+2} periods and so forth until all players of generation $k-1$ have left the game. After this the overlap phase ends and $T-K$ periods in the normal phase follow where no player will exit and hence no messages will be received. The overlap phase will then restart and K_1 periods later the player $1(k)$ will exit, sending a message to his successor. K_2 periods later $2(k)$ will exit and send a message and so on until agent $i(k)$ finally exits the game himself. In all $i(k)$ will therefore receive N messages.

The messages received by $i(k)$ are denoted

$$m_{i(k-1)}, m_{i+1(k-1)}, m_{i+2(k-1)}, ...m_{N(k-1)}, m_{1(k)}, ..., m_{i-1(k)}$$

Where the subscript indicates the sender. We incorporate these messages into $i(k)$'s *history* denoted $\overline{h}^t_{i(k)}$, which is given by

$$\overline{h}^t_{i(k)} = (m_{i(k-1)}, m_{i+1(k-1)}, .., m_j, h^t_{i(k)})$$

Here j is the most recent message received and $N \geq j \geq 1$. The full set of $i(k)$'s histories in period t, $\overline{H}^t_{i(k)}$ is now given by $\overline{H}^t_{i(k)} = M^j \times H^t_{i(k)}$ given j messages have been received. Note that M is the message space and that each message is an element of this such that $m \in M$.

There are two messages that can be sent by outgoing agents. The first is m^0, indicating that no organization has deviated from the equilibrium strategy, the other is the message m^i which indicates that organization i has deviated, where $i \in N$. The choice of which message to send is associated with various costs, with the cost to organization i of sending m^j given by $c^j_i \geq 0$. The precise value of c^j_i for all i and for all j will be determined alongside the other parameters of the model. The cost of m^0 for all organizations is given by 0, that is $c^0_i = 0$ for all i. This assumption represents that sending m^0 in fact corresponds to sending no message, which we assume is costless.

Given the above a strategy profile in the OLG game denoted (g, μ) is made up of a profile of both behavioral and reporting strategies of the players. The behavioral strategy of $i(k)$ maps from the set of player $i(k)$'s histories at period t to his strategies in the stage game, $g_{i(k)} : \overline{H}^t_{i(k)} \rightarrow \Delta(A_i)$. The reporting strategy of a player $i(k)$ maps from his history after his final period, denoted κ, to the set of messages, $\mu_{i(k)} : \overline{H}^\kappa_{i(k)} \rightarrow M$. The strategy profile $(g, \mu) = \{(g_{1(k)}, \mu_{1(k)}), (g_{2(k)}, \mu_{2(k)}), ..., (g_{N(k)}, \mu_{N(k)})\}$ therefore represents an equilibrium if and only if no player $i(k)$ has an incentive to deviate from it given any history, $\overline{h}^{t}_{i(k)}$. Such an equilibrium strategy is constructed below for the folk theorem.

3 Folk Theorem and Proof

In the OLG game defined above the following folk theorem obtains.

Theorem 1 *OLG Folk Theorem with N Organizations: Suppose that the stage game $G = \langle N, (A_i), (u_i) \rangle$ is of full dimension, i.e., $int(V) \neq \emptyset$ where $int(V)$ is the interior of V. For all $v^* \in int(V)$ and all $\delta \in (0, 1)$, there exists $\varepsilon > 0$ and there exists $\underline{T}$ such that for all $T \geq \underline{T}$, v^* is attained by a strongly stationary, perfect public, sequential equilibrium of this OLG game.*

Proof To prove the above folk theorem holds in the OLG game model outlined the equilibrium strategy denoted (g^*, μ^*) must compel agents to play the equilibrium strategy in the stage game each period and then to send a truthful message upon retirement. This is difficult because players face different incentives depending on whether they are *young* or *old* and whether there is a player from a different organization who is old. We denote an *old* agent here as one with P or fewer periods remaining before he exits the game and we denote as *young* all those players with fewer than P periods left before they exit the game. Given this only one agent can be old in any given period as by assumption we define K_i such that $K_i > P$.

Thus there are three cases a player $i(k)$ can find himself in:

1. He is young and all other players are young
2. He is young and there is one old player
3. He is old and all other players are young

Given this the equilibrium strategy dictates the following path of play. Each period every $i(k)$ plays a strategy α_i^*, together these strategies attain the profile α^* and $i(k)$ attains a payoff of v_i^*. After his last stage game, $i(k)$ then additionally sends the signal m^0, indicating that no organization has deviated from the equilibrium path. The v_i^* above is the payoff to $i(k)$ from the payoff profile v^*. Notice that the payoff profile v^* in the folk theorem is defined such that $v^* \in int(V)$. The interior of V here consists of those payoff profiles with a neighborhood of profiles within V of radius $\varepsilon > 0$. Further note that by assumption $int(V) \neq \emptyset$.

Naturally an opportunity might exist for a player $i(k)$ to unilaterally deviate from the above path of play if there are no contingencies built into the equilibrium strategy. For this reason the equilibrium strategy addresses each of the three possible scenarios where $i(k)$ can deviate, that is when he is in cases 1, 2 and 3.

We define $d = \max_i d_i$, where $d_i = \max_{\alpha,\alpha'}(u_i(\alpha) - u_i(\alpha'))$, as the maximum gain of any player from a deviation from the equilibrium strategy in the stage game. It will be shown below that players will not deviate from the equilibrium strategy, however there is one caveat, that is the parameters in the OLG game, which must be integers and positive, must be chosen to satisfy the bounds below.

The last three of these parameters, c^i_j, K_i and T have been mentioned above. Respectively, they refer to the cost of organization j sending m^i, the number of periods in the overlap phase between $i-1(k)$ exiting the game and $i(k)$ exiting the game, and the length of each players lifetime in the game.

The first four parameters will now be briefly defined. In the equilibrium strategy, P' is the number of periods of minimaxing some $i(k)$ if he deviates when all the players are young. P'' is the number of periods of rewarding the $N-1$ other players, after minimaxing $i(k)$. This bound is found by looking at only players minimaxing $i(k)$ that receive a lower payoff from doing so than v^*_j. This is further outlined below, and note that $P = P' + P''$.

We define Q'_i as the number of periods $i(k)$ is minimaxed if m^i is received by players. Q''_i is the number of periods the other players receive reward payoffs for carrying out the minimaxing of $i(k)$ in this case. Because the number of minimax and reward periods must be large enough to deter deviations and to make minimaxing deviators rational, the parameters of the OLG game, c^i_j, K_i and T, must be carefully chosen such that enough periods exist for these minimaxing and reward phases to be implemented. To ensure this, bounds have been derived from the equilibrium strategy for these parameters. Given some stage game G, by simply choosing values that fulfil the bounds, we can ensure that the OLG game will be such that equilibrium strategy will hold.

Given that, the bounds on $P', P'', Q'_i, Q''_i, c^i_j, K_i$ and T which must be hold for all $i, j \in N$ where $i \neq j$ are:

$$P' > \frac{d}{\delta(v^*_i - \underline{u}_i)} \forall i \text{ (Bound A)}$$

$$P'' > \frac{d}{w^j_i - v^*_i} + P' \frac{\underline{u}_i - u_i(\underline{\alpha}^j)}{w^j_i - v^*_i} \; \forall i, j \text{ s.t. } u_i(\underline{\alpha}^j) \leq v^*_i \text{ (Bound B)}$$

$$Q'_i > PP' \forall i \text{ (Bound C)}$$

$$Q'_i(u_j(\underline{\alpha}^i) - v^*_j) + Q''_i(w^i_j - v^*_j) = \frac{c^i_j}{\delta} \forall i, j \text{ (Bound D)}$$

$$K_i > Q_j + P \qquad \forall i, j \text{ (Bound E)}$$

$$T \geq K = K_1 + K_2 + ... + K_N \text{ (Bound F)}$$

Note that the parameter being chosen is on the left hand side of the expression except for the case of bound D where Q''_i and c^i_j are chosen concurrently to fulfil the expression for all combinations of $j \in N$ and $i \in N$ where $j \neq i$. Further note that c^i_j need not be an integer but must be non-negative. Also, bound B on P'' is found by looking at only those i for whom sanctioning j give them a payoff below their equilibrium payoff and thus $u_i(\underline{\alpha}^j) \leq v^*_i$. This is done because those other players for which

$u_i(\underline{\alpha}^j) > v_i^*$ also have $w_i^j = v_i^*$, and including these players when choosing bound B would make the bound undefined.

Case 1 Here a deviation from α_i^* by $i(k)$ is responded to by the other $N-1$ players choosing to minimize $i(k)$'s maximum payoff for P' periods. In this way the other players *sanction* $i(k)$. This will give $i(k)$ a payoff below v_i^* as $v \in int(V)$. This is done to reduce $i(k)$'s payoff, unfortunately during the sanctioning of $i(k)$ other players may receive payoffs under their equilibrium payoff v_j^*. These players will require a reward to induce them to sanction i. Of course some players may receive more than their equilibrium payoff while sanctioning and these players will require no reward as they're happy to do the job.

As such, after P' periods have passed a reward period is played for P'' periods where players choose a strategy profile that attains a *reward* payoff profile denoted $w^i \in V$, where $i(k)$ was the deviator. The payoffs to each player $i \in N$ in this profile are denoted as w_j^i and are as follows; $w_i^i = v_i^*$ for the deviator $i(k)$, also $w_j^i = v_j^*$ for players where $u_j(\underline{\alpha}^i) > v_j^*$, so players for whom sanctioning $i(k)$ increases their payoffs, and a payoff of $w_j^i > v_j^*$ for players where $u_j(\underline{\alpha}^i) \leq v_j^*$, so for players for whom sanctioning $i(k)$ lowers their payoffs. Note that in the equilibrium strategy the original deviator does not require a reward payoff greater than v_i^* to prevent further deviations as while he is being sanctioned he is expected to play his best response and by definition has no incentive to deviate from this. We know that the profile w^i exists as $v \in int(V)$, so $w^i \in V$ exists.

This indicates that for $i(k)$ to be constrained not to deviate from the usual equilibrium path of play, his maximal payoffs from deviating must be less than his payoffs from playing according to the equilibrium strategy. For this to be the case, the following incentive constraint must hold, where the payoffs for the next $P = P' + P''$ periods are contrasted for the cases of $i(k)$ deviating and $i(k)$ not deviating.

$$d_i + P'\underline{u}_i + P''v_i^* < Pv_i^* \tag{1}$$

And 1 does hold as it can be derived from bound A.

The above incentive constraint prevents a young agent choosing to deviate but the sanctioners must weigh up their alternatives as well. First note that if $j(k)$ is constrained from deviating from his normal payoff path by (1) then he will not deviate from sanctioning the deviator if $u_j(\underline{\alpha}^i) > v_j^*$ and thus raises his payoffs. On the other hand if $u_j(\underline{\alpha}^i) \leq v_j^*$ and sanctioning lowers his payoffs then during the reward phase the sanctioner will not deviate if (1) holds as he receives above his normal payoff. Further as the number of future periods of sanctioning declines while minimaxing the deviator his equilibrium strategy payoffs over the next P periods increase. So he will deviate in the first period he must sanction the deviator, or not at all. In that period the incentive constraint that must hold for the equilibrium strategy to be chosen is

$$d_j + P'\underline{u}_j + P''v_j^* < P'u_j(\underline{\alpha}^i) + P''w_j^i \tag{2}$$

This can be derived from

$$d_j < P'[u_j(\underline{\alpha}^i) - \underline{u}_j] + P''[w^i_j - v^*_j] \tag{3}$$

Which in turn can be derived from bound B.

As will be outlined later old agents must send truthful messages in equilibrium. If the message m^i is sent this denotes organization i deviated. The equilibrium strategy then dictates that $i(k)$ will be minimaxed for Q'_i periods and then w^i will be played for Q''_i periods, before play reverts to the usual continuation. Analogously to the scenario explored above, players for whom sanctioning $i(k)$ is its own reward have no incentive to deviate from this path. Again though for players for which sanctioning $i(k)$ hurts, they must be constrained from deviating in the first period of sanctioning $i(k)$. Observing $Q_i = Q'_i + Q''_i$ is such that $Q_i > P$, then for the payoffs of not deviating to be more attractive here the following must hold,

$$d_j + P'\underline{u}_j + P''v^*_j + [Q_i - P]v^*_j < Q'_i u_j(\underline{\alpha}^i) + Q''_i w^i_j \tag{4}$$

(4) can be derived from the following

$$d_j < P'[v^*_j - \underline{u}_j] + Q'_i[u_j(\underline{\alpha}^i) - v^*_j] + Q''_i[w^i_j - v^*_j] \tag{5}$$

Which in turn is derived from bound A and bound D.

Case 2 Here to prevent a young agent deviating an immediate P period punishment strategy cannot be invoked as there are fewer than P periods left before the old agent exits the game. The equilibrium strategy therefore has the players play some Nash equilibrium of the stage game each period until the old agent retires, the old agent upon retirement sends the message m^i indicating organization i deviated and i is then minimaxed for Q'_i periods which is followed by Q''_i periods of the reward payoff profile w^i. At most then a deviator can increase his payoff by d_i for P periods before being punished. Thus for $i(k)$ to have no incentive to deviate the following must hold

$$Pd_i + Q'_i\underline{u}_i + Q''_i v^*_i < Q_i v^*_i \tag{6}$$

We can derive (6) from

$$Pd_i < Q'_i[v^*_i - \underline{u}_i] \tag{7}$$

(7) holds as it can be derived from bound A and bound C.

Note that no player has an incentive to deviate from the Nash equilibrium by definition, and that the punishment of $i(k)$ will occur as after m^i is received all agents are young and case 1 holds.

Case 3 An old agent must be dissuaded from deviating both from the equilibrium strategy in the stage game and from sending a truthful message when he exits. The first of these tasks involves understanding what occurs in the equilibrium strategy if an old player $i(k)$ deviates in the stage game. If $i(k)$ deviates a Nash equilibrium will be played until he exits the game as before. Now however the message he sends, which is truthful in equilibrium, is m^0. This signifies that no other organization has deviated while mentioning nothing about his own deviation.[3] Following this the agents play α^* in the stage game until the next agent retires. This agent then sends the message m^i and the successor of the deviator will be minimaxed for Q'_i periods which will be followed by a Q''_i period reward phase where w^i is played.

The players will not deviate from this path in the Nash equilibrium stage as by definition it is a best response. Neither will any player have an incentive to deviate after the deviator retires and α^* is played, as they will be in one of the three cases and receive the usual payoff v_i^*. Players also have no incentive to deviate during the eventual punishment of the deviator's organization, as at that point all players will be young.

Given this an old player weighs up the gain from deviating for at most P periods and increasing his payoff in the stage game, against the loss of payoffs which his successor receives due to being sanctioned, which he evaluates at a discounted rate. Therefore in the first period of being old, when he could gain the most from deviating, the following incentive constraint must hold for each player $j = 1, 2, 3...N-1$ but not $j = N$.

$$Pd_j + \delta[K_{j+1}v_j^* + Q'_j\underline{u}_j + Q''_j v_j^*] < \delta[K_{j+1}v_j^* + Q_j v_j^*] \tag{8}$$

In the case of player N there is no player $N+1$ so he must wait $T - K + K_1$ periods before the next player after him retires. That is player 1. So for player $j = N$ the following constraint must hold

$$Pd_j + \delta[(T - K + K_1)v_j^* + Q'_j\underline{u}_j + Q''_j v_j^*] < \delta[(T - K + K_1)v_j^* + Q_j v_j^*]$$

Both of these equations can be derived from the following

$$Pd_j < \delta Q'_j[v_j^* - \underline{u}_j] \tag{9}$$

In turn 9 can be derived from bound A and bound C.

Old players upon leaving the game must be constrained to send a truthful message. This means sending m^i if organization i deviated and sending m^0 if no other organization deviated. To make sure a truthful message is always sent the payoffs from both m^i and m^0 must be the same for $j(k)$. Recall that $j(k)$ discounts the payoffs of his successor and that sending m^i indicates that his successor will receive the payoffs from punishing i then being rewarded, whereas by sending m^0 $j(k)$'s successor will receive the normal payoff of v_i^* instead. Further note that in the case of a player for whom sanctioning is its own reward $w_j^i = v_j^*$. So for truthful messages to be sent the following must hold

$$\delta Q_i v_j^* = \delta[Q'_i u_j(\underline{\alpha}^i) + Q''_i w_j^i] - c_j^i \tag{10}$$

This can be derived from bound D.

Finally note that by setting $T \geq K$, we can ensure that the OLG model is set up so that players lives are sufficiently long for the overlap phase to be constructed as outlined. If $T > K$ is chosen then the additional $T - K$

[3] Even in OLG games players have the right to remain silent!

periods will become part of the normal phase. Further notice that as $K_i > Q_j + P$ for all j the overlap phase will involve players that are young and old while in the normal phase players are only ever young.

As no player has any incentive to deviate from the equilibrium strategy and every period the payoff profile $v^* \in int(V)$ will be attained the folk theorem holds. □

Remark 1 In the above folk theorem, it is meant by strongly stationary that the payoffs to every agent in an organization i are in fact the same for every period. This follows Lagunoff and Matsui[6]. Further notice that the equilibrium concept is that of a perfect public sequential equilibrium. By this it is meant that the folk theorem in the OLG game relies on equilibrium strategies wherein the players condition their stage game strategies on only publicly available information. Also note that players limited information of past play in the game means that the equilibrium must be sequential. Notice though that because it is a PPE there are shared beliefs in what constitutes the equilibrium play in the stage game.

4 A Numerical Example

One example of the OLG game presented above can be found in a price fixing game amongst an oligopoly of three firms. Each firm is run by a manager who's payoffs are his salary, which is a proportion of the profits of his organization during his tenure. After retirement he receives the same salary as his successor, although it is discounted by δ. This represents his original contract with the firm including shares of that firm which cannot be disposed of until his successor retires.

A retiring manager communicates publicly with his successor, although informing him that another firm deviated is costly. After all informing his successor that something significant occurred during his tenure requires time and effort, but saying nothing, and letting his successor believe nothing occurred is costless.

The game is a form of prisoner's dilemma, where if the firms collude and set high prices they all benefit. However each firm then has an incentive to set low prices and undercut the others.

	Left	Right
Up	12,12,12	5,16,5
Down	16,5,5	9,9,5

A

	Left	Right
Up	5,5,16	5,9,9
Down	9,5,9	8,8,8

B

Fig. 3 The game in normal form. Player 1 chooses the row, player 2 chooses the column, and player 3 chooses the table.

In the OLG game we will set $v_1^* = v_2^* = v_3^* = 10$. If we assume $\varepsilon = 0.5$ then $v^* \in int(V)$. We now let $w_j^i = 11 \forall, i, j$ where $j \neq i$. The stage game then implies the players can receive the following payoffs:

For all players	$d = 7$
Player 1	$\underline{u}_1 = 8$ $w_1^2 = 11$ $w_1^3 = 11$ $u_1(\underline{\alpha}^2) = 8$ $\underline{u}_1(\underline{\alpha}^3) = 8$
Player 2	$\underline{u}_2 = 8$ $w_2^1 = 11$ $w_2^3 = 11$ $u_2(\underline{\alpha}^1) = 8$ $\underline{u}_2(\underline{\alpha}^3) = 8$
Player 3	$\underline{u}_3 = 8$ $w_3^1 = 11$ $w_3^2 = 11$ $u_3(\underline{\alpha}^1) = 8$ $\underline{u}_3(\underline{\alpha}^2) = 8$

The players payoffs then, are as follows.
INSERT 3D PAYOFF SPACE
The folk theorem implies that $v^* \in int(V)$ can be attained each period of this game. For the equilibrium strategy to hold though, the parameters it uses must be chosen to satisfy Bounds A to F. This can be ensured by simply working through them. One set of choices that satisfy the equilibrium strategies requirements are given below.

Bound A	$P' = 8$
Bound B	$P'' = 8$
Bound C	$Q_1' = 130$ $Q_2' = 130$ $Q_3' = 130$
Bound D	$Q_1'' = 270$ $c_1^2 = 5$ $c_1^3 = 5$ $Q_2'' = 270$ $c_2^1 = 5$ $c_2^3 = 5$ $Q_3'' = 270$ $c_3^1 = 5$ $c_3^2 = 5$
Bound E	$K_1 = 420$ $K_2 = 420$ $K_3 = 420$
Bound F	$T = 1300$

Given these choices of K_i and T, the overlapping generations are as follows.

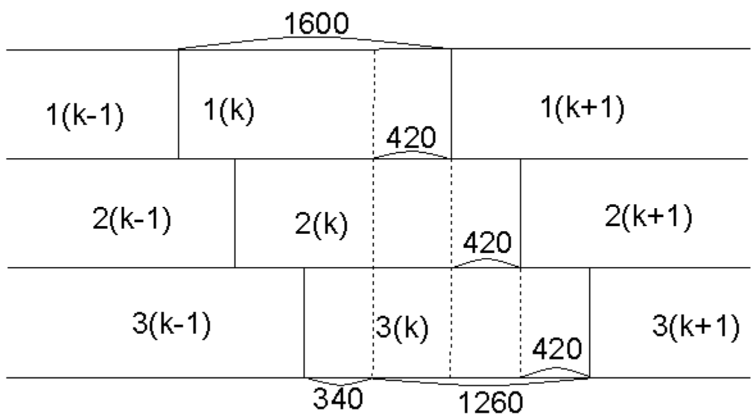

Fig. 4 The OLG game chosen above.

Utilizing the equilibrium strategy with these parameters the folk theorem indicates v^* is attained each period of the stage game. Each players payoff is therefore the $T = 1600$ payoffs from his time as manager, plus the discounted 1600 payoffs of his successor. In total a manager receives a payoff of 24,000 which is considerably higher than the 19,200 he would receive if all players chose their strictly dominant strategy in the stage game, and received 8 every period. The folk theorem implies the oligopoly can sustain higher profits using the equilibrium strategy when its bounds are met, as was the case with the choices made above.

5 Imperfect Communication

The above model assumed that messages from outgoing managers were perfectly observed by other players. We feel that this is a restrictive assumption to make as miscommunication from the outgoing agent may be possible. As such we weaken the information an incoming agent has in the game by assuming a message is only correctly observed with some probability. Again, a folk theorem obtains in this case although it is no longer strongly stationary result.

The above OLG game can be generalized by assuming that a message $m \in M$ sent by an outgoing player is correctly observed only with some probability $p \in [0, 1]$. This means that with probability $1 - p$ an incorrect message will be observed. There are $N - 1$ possible incorrect messages each of which is observed with equal probability given the correct message does not arrive. These are the N messages the player did not send with the message m^i ruled out as i cannot indict himself. Thus the probability of some message $m' \in M$ arriving when $m \in M$ is sent, denoted $p(m'|m)$, is given by the following

- $p(m^0|m^0) = p(m^i|m^i) = p$ for all $i \in N$
- $p(m^0|m^i) = p(m^i|m^0) = p(m^j|m^i) = \frac{1-p}{N-1}$ for all $i \in N$and all $j \in N$

Given that the above above changes the model to one of imperfect communication, the following folk theorem obtains.

Theorem 2 *Folk Theorem with N Organizations and Imperfect Communication: Suppose that the stage game $G = \langle N, (A_i), (u_i) \rangle$ is of full dimension, i.e., int$(V) \neq \emptyset$ where int(V) is the interior of V. For all $v^* \in$int(V) and all $\delta \in (0, 1)$, there exists $\varepsilon > 0$ and for all $p \in (\frac{1}{N}, 1]$ and for some $q > 0$, there exists $\underline{T}$ such that for all $T \geq \underline{T}$, v a payoff arbitrarily close to v^* is attained as the average stage game payoff of an imperfect public, sequential equilibrium of this OLG game, such that $\|v^* - v\| < q$.*

Proof This folk theorem uses the equilibrium strategy outlined above and the bounds on the parameters are exactly those given earlier, with some exceptions. These changes are a new bound for p, the probability of the correct message being observed, the bound on P' is changed and there is a further bound given on T. Finally we also define Q as $Q = max_i Q_i$. These changes are given in the following and apply for all $i \in N$

$$p \in (\frac{1}{N}, 1] \quad \text{(Bound G)}$$

$$P' > \frac{(N-1)d}{(Np-1)\delta(v_i^* - \underline{u}_i)} \qquad \forall i \in N \text{ (Bound A*)}$$

$$T > \frac{NQd - c_i^j}{q}\sqrt{N} \qquad \forall j \in N \quad \text{(Bound H)}$$

It is interesting to note that under bound G the probability of the true message being observed is p which cannot ever fall as low as $\frac{1}{N}$. Notice that if $p = \frac{1}{N}$ were chosen that the probability of each of the $N-1$ other messages being observed would become $\frac{1-\frac{1}{N}}{N-1}$, that is, $\frac{1}{N}$. So by restricting p to $p \in (\frac{1}{N}, 1]$, we in fact restrict the probability of the message sent being observed to being greater than the probability of each of the messages not sent being observed. Intuitively this seems reasonable as to ensure that players send a truthful message in equilibrium, they condition their decision on the fact that the message chosen is slightly more likely to be observed than any other. The three cases outlined earlier will now be examined in turn below.

Case 1 Once some message is received play is dictated by it. As such there is no need to recheck incentives for players in case 1. Also note that all the original incentive constraints can still be derived from the bounds as before except for (1) which can now be derived from bound A*. So case 1 is relatively unaffected by the changes to the model.

Case 2 As previously in this case after an agent $i(k)$ deviates a Nash equilibrium is played until the old player $j(k)$ leaves the game. The old player then elects to send m^i however now m^i is observed only with probability p. With probability $\frac{1-p}{N-1}$, each of the other $N-1$ possible messages denoted m^b will be observed. Note that $b = 0, 1, ..., N$ although $b \neq i$ and $b \neq j$. On the other hand if $i(k)$ does not deviate then with probability p the message m^0 is observed and $i(k)$ receives the usual equilibrium payoffs. Again with probability $\frac{1-p}{N-1}$ each of the incorrect messages denoted m^b is observed and now $b = 1, 2, ..., N$ although $b \neq i$ and $b \neq j$. Note that m^0 cannot be received as an *incorrect* message in distinction to the above case, because here m^0 is the *correct* message. To have no incentive to deviate then the following must now hold for $i(k)$

$$Pd_i + p[Q'_i\underline{u}_i + Q''_i v_i^*] + \frac{(1-p)}{N-1}Q_i v_i^* + \frac{(1-p)}{N-1}\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b] <$$
$$pQ_i v_i^* + \frac{(1-p)}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] + \frac{(1-p)}{N-1}\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b] \tag{11}$$

Note that (11) can be derived from

$$Pd_i < \frac{Np-1}{N-1}Q'_i(v_i^* - \underline{u}_i) \tag{12}$$

And (12) can be derived from bound A*, bound G and bound C.

Case 3 Recall that on the equilibrium path if a player $j(k)$ deviates while old he is then expected to send the message m^0 to his successor. Following this the next player to leave the game is expected to send the message m^j which initiates the sanctioning of $j(k)$'s successor. Each of these messages is correctly observed only with probability p. Further it is assumed the cost of the message c_i^j that a player i pays now depends on the message observed, not the message sent. So even if m^0 is sent, if m^j is observed i pays c_i^j. Noting that it will be shown below that in equilibrium a player $j(k)$ sends a truthful message m^0 upon retirement, whether or not he has deviated, the payoffs to his successor are the same until the next message is sent. Therefore the incentive constraint that prevents $j(k)$ from deviating refers to the increased payoffs he receives before exiting the game, and the lower payoffs his successor receives after the next player leaves the game and m^j is sent. For $j(k)$ to have no incentive to deviate the following must then hold

$$Pd_j + \delta[p(Q'_j\underline{u}_j + Q''_j v_j^*) + \frac{(1-p)}{N-1}Q_j v_j^* + \frac{(1-p)}{N-1}\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b]] <$$
$$\delta[pQ_j v_j^* + \frac{(1-p)}{N-1}(Q'_j\underline{u}_j + Q''_j v_j^*) + \frac{(1-p)}{N-1}\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b]] \tag{13}$$

Further (13) can be derived from

$$Pd_j < \delta\frac{Np-1}{N-1}Q'_j(v_j^* - \underline{u}_j) \tag{14}$$

And (14) holds as it can be derived from bound B*, bound G and bound C.

The old player must be constrained to send a truthful message in equilibrium. This means that his payoff from both m^0 and m^i for all $i \in N$ must be the same. This is similar to the original case however here again the probabilities of the correct and incorrect messages being observed must be included. The equality that shows $j(k)$ is indifferent between sending any of the messages and hence means he has no incentive to deviate from sending the truthful message, is given by

$$\begin{aligned} p\delta Q_i v_j^* + \frac{(1-p)}{N-1} \sum_{b=1, b\neq j}^{N} (\delta(Q'_b u_j(\underline{\alpha}^b) + Q''_b w_i^b) - c_j^b) \\ = p[\delta(Q'_i u_j(\underline{\alpha}^i) + Q''_i w_j^i) - c_j^i] + \frac{(1-p)}{N-1} \delta Q_i v_j^* \\ + \frac{(1-p)}{N-1} \sum_{b=1, b\neq j, b\neq i}^{N} (\delta(Q'_b u_j(\underline{\alpha}^b) + Q''_b w_i^b) - c_j^b) \end{aligned} \tag{15}$$

This can be derived from

$$\delta Q_i v_j^* = [\delta(Q'_i u_j(\underline{\alpha}^i) + Q''_i w_j^i) - c_j^i] \tag{16}$$

Which is the same as the original message constraint, equation (10), and can thus be derived from bound D.

It has been shown then that no player ever has an incentive to deviate from the equilibrium strategy.

Payoffs Players payoffs must now be shown to be within q of v^*, the payoffs being approximated. To show this denote some feasible payoff to i as n_i. We then denote Q as the longest punishment phase of any player. That is $Q = max_i Q_i$. Let v_i denote the average payoff attained by i in his lifetime. The average payoff v_i is the sum of the $T - NQ$ periods of v_i^* the equilibrium payoff and NQ periods of n_i, all divided by T, a players total lifetime. We define v_i in this way because with some probability $1 - p$ an incorrect message will be observed and v_i^* will not necessarily be attained for the first Q periods after that message. In total a player may not receive v_i^* for up to NQ periods out of his total lifetime T. So for NQ periods, a players payoff is n_i which can differ from v_i^*. Also note that some message costing c_i^j will be sent. With this in mind the average payoff to player i is $v_i = \frac{(T-NQ)v_i^* + NQn_i - c_i^j}{T}$. Given this it can be shown as stated in the folk theorem that the average payoff profile is within q of the payoff profile being approximated, that is

$$\|v^* - v\| < q \tag{17}$$

This condition is true even under each player's worst case scenario. That is when they attain the lowest payoffs possible along the folk theorems equilibrium path.

As equation (17) can be derived from bound H. That completes the proof of the folk theorem for some $\delta \in (0, 1)$ and $\varepsilon > 0$. □

6 A Numerical Example with Imperfect Communication

Consider the same OLG game presented earlier but with outgoing managers only able to correctly communicate the signal they want to with probability p. This represents that a public statement from the outgoing manager can be misinterpreted by the players. The exiting player is charged the cost of the message received by the players because under his contract his organization can recoup any expenses he makes from him, and upon mistakenly receiving the message that some organization deviated, they charge him the cost of preparing that message. This scenario accords with the imperfect communication OLG game outlined above.

The folk theorem implies that some $v^* \in int(V)$ can be attained each period of this game. And again we set $v_1^* = v_2^* = v_3^* = 10$. So if $\varepsilon = 0.5$ this $v^* \in int(V)$. We further assume that $q = 0.75$. The equilibrium strategy will hold here if we choose parameters that satisfy Bounds A* through to H. This is the case if we again simply work through them. Choices that satisfy the equilibrium requirements are given below.

Bound G	$p = \frac{1}{2}$
Bound A*	$P' = 30$
Bound B	$P'' = 8$
Bound C	$Q_1' = 1200$ $Q_2' = 1200$ $Q_3' = 1200$
Bound D	$Q_1'' = 2410$ $c_1^2 = 5$ $c_1^3 = 5$ $Q_2'' = 2410$ $c_2^1 = 5$ $c_2^3 = 5$ $Q_3'' = 2410$ $c_3^1 = 5$ $c_3^2 = 5$
Bound E	$K_1 = 3700$ $K_2 = 3700$ $K_3 = 3700$
Bound F and H	$T = 176000$

These parameters ensure that the equilibrium strategy will be followed and each player will on average attain a payoff of 9.9967 during his tenure. The value of T in this particular example is quite high. With another stage game this of course could be lower, and with other choices for the other parameters this could also be lower. In fact note that if only bound F had to hold a T greater than 11,100 periods would have been sufficient. However bound H implied a T above 175,064 periods was necessary. Of course note that as time is discrete in this model, a managers tenure of $T = 176000$ periods could equate to hours, months or years. It is entirely arbitrary.

As the folk theorem holds, it corresponds to a total payoff, being a players payoff plus his discounted successor's payoffs, of 2,639,115 on average. This is considerably higher than the 1,707,997.5 total payoff each player would receive if the nash equilibrium were played each period and m^0 was the message always sent. So again, even with the possibility of the truthful message being misinterpreted with probability $p = \frac{1}{2}$, the oligopoly can sustain higher payoffs using the bounds that obtain the folk theorem.

7 Conclusion

In this paper we have achieved some generalizations of the folk theorem of Lagunoff and Matsui ([6]). These have been to heterogenous players, N organizations and imperfect communication.

This implies that cooperation can indeed be sustained in oligopolies and similar groupings of long run organization controlled by short run agents. Although these results rely on the equilibrium strategies outlined, which in turn rely on players lifetimes, T, and the overlaps, K_i, being of sufficient length. Further, our results rely on communication channels being opened between outgoing agents and their successors and that agents have an interest in the outcomes achieved by their successor. This suggests that paying people in such a way as to induce an interest in the long run outcome of their companies may be very useful in terms of the continued cooperation of the cartel. While some of these points may seem obvious, the result is not. After all if a manager was made better off by lying to his successor and whitewashing the past, he would do so and the folk theorem would break down. As such the equilibrium strategy had to carefully balance the payoffs from lying with those from telling the truth. This became even more interesting when even choosing the truthful message did not necessarily imply it would be received in the second folk theorem.

In the above folk theorem we believe equation (17) can be attained with a less restrictive condition than bound H. Bound H assumes that for NQ periods *any* payoff may be attained. In fact the payoffs attained will not deviate by more than a limited amount from v_i^*. So bound H specifies T must be greater than it has to be. Given this a lower bound on T could be derived, although we have not done so.

Naturally the question of possible further extensions and work on OLG games also arises. The work of Morris and Mailath [7] certainly suggests itself as one avenue of development. They have worked on repeated private monitoring games with imperfectly observed signals sent to each player every period, and their folk theorem details the circumstances under which imperfect private monitoring games can approximate the outcomes of their perfect public counterparts. An appealing aspect of this line of development is that Morris and Mailath require that the games they examine have a structure whereby only a finite history is needed for players to condition their actions. This fits naturally into the OLG folk theorems' use of messages. It also would be interesting to see if private monitoring results such as these could be carried across to the OLG framework.

It may also be possible that the model used in the second folk theorem could be extended from players choices over which message to send to players' choices over a distribution of messages to send. Costs related to the weight of 'truthfulness' could then be used to perhaps facilitate a result. A further extension would be to generalize the nature of the overlapping generations used in the model. This could perhaps be done to take account of both differences among the organizations and of differences among the generations. That is, each individual could be assigned some unique lifetime that would not necessarily have to be T. The main question in this case seems to be what occurs when two agents simultaneously exit the game, as there must be some mechanism to determine who's message dictates future play. In fact it may be necessary to ensure that there is only one old player in any given period. Another possible approach to extending the work on folk theorems in overlapping generations games is suggested by the literature on random games[11].

Finally while development in OLG games generally follow those in repeated games more broadly something unique could be introduced by having the stage game change with each new agent entering the game. That is, the stage game is generalized so that each new agent does not have to inherit his organizations payoff function and action set. We note that it is not at all obvious how results in such a model would be found. All such future work would radically further the scope of the present paper.

A Derivation of (1)

Bound A states that

$$\begin{aligned}
& P' > \tfrac{d}{\delta(v_i^* - \underline{u}_i)} \\
\Rightarrow\ & P'\delta(v_i^* - \underline{u}_i) > d \quad \text{as } \delta(v_i^* - \underline{u}_i) > 0 \\
\Rightarrow\ & P'\delta(v_i^* - \underline{u}_i) > d \geq d_i \quad \text{as } d \geq d_i \\
\Rightarrow\ & P'\delta(v_i^* - \underline{u}_i) > d_i \\
\Rightarrow\ & P'(v_i^* - \underline{u}_i) > P'\delta(v_i^* - \underline{u}_i) > d_i \quad \text{as } \delta \in (0,1) \\
\Rightarrow\ & P'(v_i^* - \underline{u}_i) > d_i \\
\Rightarrow\ & d_i < P'(v_i^* - \underline{u}_i) \\
\Rightarrow\ & d_i < (P'v_i^* - P'\underline{u}_i) + (P''v_i^* - P''v_i^*) \\
\Rightarrow\ & d_i < (P' + P'')v_i^* - (P'\underline{u}_i + P''v_i^*) \\
\Rightarrow\ & d_i < Pv_i^* - (P'\underline{u}_i + P''v_i^*) \\
\Rightarrow\ & d_i + P'\underline{u}_i + P''v_i^* < Pv_i^*
\end{aligned}$$

Thus (1) has been derived.

B Derivation of (2)

Recall that (2) must hold only for players for whom 'sanctioning hurts', so $v_i^* \geq u_i(\underline{\sigma}^j)$ and thus $w_i^j > v_i^*$. Bound B states

$$\begin{aligned}
& P'' > \tfrac{d}{w_i^j - v_i^*} + P'\tfrac{\underline{u}_i - u_i(\underline{\alpha}^j)}{w_i^j - v_i^*} \\
\Rightarrow\ & P''(w_i^j - v_i^*) > d + P'(\underline{u}_i - u_i(\underline{\alpha}^j)) \quad \text{as } w_i^j - v_i^* > 0 \\
\Rightarrow\ & P''(w_i^j - v_i^*) - P'(\underline{u}_i - u_i(\underline{\alpha}^j)) > d \\
\Rightarrow\ & P''(w_i^j - v_i^*) + P'(u_i(\underline{\alpha}^j) - \underline{u}_i) > d \\
\Rightarrow\ & P''(w_i^j - v_i^*) + P'(u_i(\underline{\alpha}^j) - \underline{u}_i) > d \geq d_i \\
\Rightarrow\ & P'(u_i(\underline{\alpha}^j) - \underline{u}_i) + P''(w_i^j - v_i^*) > d_i \\
\Rightarrow\ & P'(u_j(\underline{\alpha}^i) - \underline{u}_j) + P''(w_j^i - v_j^*) > d_j \quad \text{as } i \text{ is arbitrary}
\end{aligned}$$

Thus (3) has been derived from the bound.

$$\begin{aligned}
&\Rightarrow d_j < P'[u_j(\underline{\alpha}^i) - \underline{u}_j] + P''[w_j^i - v_j^*] \\
&\Rightarrow d_j < P'u_j(\underline{\alpha}^i) - P'\underline{u}_j + P''w_j^i - P''v_j^* \\
&\Rightarrow d_j < [P'u_j(\underline{\alpha}^i) + P''w_j^i] - [P'\underline{u}_j + P''v_j^*] \\
&\Rightarrow d_j + P'\underline{u}_j + P''v_j^* < P'u_j(\underline{\alpha}^i) + P''w_j^i
\end{aligned}$$

Thus (2) has been derived.

C Derivation of (4) and (10)

Bound D states

$$\begin{aligned}
&Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) = \tfrac{c_j^i}{\delta} \\
&\Rightarrow \delta(Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*)) = c_j^i \\
&\Rightarrow \delta Q_i'u_j(\underline{\alpha}^i) - \delta Q_i'v_j^* + \delta Q_i''w_j^i - \delta Q_i''v_j^* = c_j^i \\
&\Rightarrow \delta Q_i'u_j(\underline{\alpha}^i) + \delta Q_i''w_j^i = \delta Q_i'v_j^* + \delta Q_i''v_j^* + c_j^i \\
&\Rightarrow \delta Q_i'u_j(\underline{\alpha}^i) + \delta Q_i''w_j^i - c_j^i = \delta(Q_i' + Q_i'')v_j^* \\
&\Rightarrow \delta(Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i) - c_j^i = \delta Q_i v_j^*
\end{aligned}$$

Notice equation (10) has been derived.

Note that Bound A states $d_i < P'(v_i^* - \underline{u}_i)$. Thus

$$\begin{aligned}
&Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) = \tfrac{c_j^i}{\delta} \\
&\Rightarrow Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) \geq 0 \quad \text{as } \tfrac{c_j^i}{\delta} \geq 0 \\
&\Rightarrow Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) + P'(v_j^* - \underline{u}_j) > d_j \quad \text{as } d_j < P'(v_j^* - \underline{u}_j)
\end{aligned}$$

Thus equation (5) has been derived.

$$\begin{aligned}
&\Rightarrow d_j < P'(v_j^* - \underline{u}_j) + Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) \\
&\Rightarrow d_j < Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) + P'(v_j^* - \underline{u}_j) \\
&\Rightarrow d_j < Q_i'(u_j(\underline{\alpha}^i) - v_j^*) + Q_i''(w_j^i - v_j^*) + P'(v_j^* - \underline{u}_j) + P''(0) \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) - Q_i'v_j^* + Q_i''w_j^i - Q_i''v_j^* + P'v_j^* - P'\underline{u}_j + P''v_j^* - P''v_j^* \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i - P'\underline{u}_j - P''v_j^* - Q_i'v_j^* - Q_i''v_j^* + P'v_j^* + P''v_j^* \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i - P'\underline{u}_j - P''v_j^* - (Q_i' + Q_i'')v_j^* + (P' + P'')v_j^* \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i - P'\underline{u}_j - P''v_j^* - Q_i v_j^* + P v_j^* \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i - P'\underline{u}_j - P''v_j^* - (Q_i - P)v_j^* \\
&\Rightarrow d_j < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i - (P'\underline{u}_j + P''v_j^* + (Q_i - P)v_j^*) \\
&\Rightarrow d_j + P'\underline{u}_j + P''v_j^* + (Q_i - P)v_j^* < Q_i'u_j(\underline{\alpha}^i) + Q_i''w_j^i
\end{aligned}$$

Thus (4) has been derived.

D Derivation of (6)

Bound A and bound C state together that

$$
\begin{aligned}
&Q'_i > PP' = PP' > P[\tfrac{d}{\delta(v_i^* - \underline{u}_i)}] \\
&\Rightarrow Q'_i > P[\tfrac{d}{\delta(v_i^* - \underline{u}_i)}] \\
&\Rightarrow Q'_i \delta(v_i^* - \underline{u}_i) > Pd \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd \quad \text{as } i \text{ is arbitrary} \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd \geq Pd_j \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd_j
\end{aligned}
$$

Notice (7) has been derived.

$$
\begin{aligned}
&\Rightarrow Pd_i < Q'_i[v_i^* - \underline{u}_i] \\
&\Rightarrow Pd_i < Q'_i[v_i^* - \underline{u}_i] + Q''_i[v_i^* - v_i^*] \\
&\Rightarrow Pd_i < Q'_i v_i^* - Q'_i \underline{u}_i + Q''_i v_i^* - Q''_i v_i^* \\
&\Rightarrow Pd_i < Q'_i v_i^* + Q''_i v_i^* - Q'_i \underline{u}_i - Q''_i v_i^* \\
&\Rightarrow Pd_i < Q_i v_i^* - [Q'_i \underline{u}_i + Q''_i v_i^*] \\
&\Rightarrow Pd_i + Q'_i \underline{u}_i + Q''_i v_i^* < Q_i v_i^*
\end{aligned}
$$

Thus (6) has been derived.

E Derivation of (8)

Bound A and bound C taken together imply

$$
\begin{aligned}
&Q'_i > PP' = PP' > P[\tfrac{d}{\delta(v_i^* - \underline{u}_i)}] \\
&\Rightarrow Q'_i > P[\tfrac{d}{\delta(v_i^* - \underline{u}_i)}] \\
&\Rightarrow Q'_i \delta(v_i^* - \underline{u}_i) > Pd \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd \quad \text{as } i \text{ is arbitrary} \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd \geq Pd_j \\
&\Rightarrow Q'_j \delta(v_j^* - \underline{u}_j) > Pd_j
\end{aligned}
$$

Thus (9) has been derived.

$$
\begin{aligned}
&\Rightarrow Pd_j < \delta Q'_j[v_j^* - \underline{u}_j] \\
&\Rightarrow Pd_j < \delta[Q'_j v_j^* - Q'_j \underline{u}_j] \\
&\Rightarrow Pd_j < \delta[Q'_j v_j^* - Q'_j \underline{u}_j + Q''_j v_j^* - Q''_j v_j^*]
\end{aligned}
$$

The following shows (8) holds for players $j = 1, 2, 3 ... N - 1$

$$
\begin{aligned}
&\Rightarrow Pd_j < \delta[K_{j+1} v_j^* - K_{j+1} v_j^* + Q'_j v_j^* + Q''_j v_j^* - Q'_j \underline{u}_j - Q''_j v_j^*] \\
&\Rightarrow Pd_j < \delta[K_{j+1} v_j^* + Q_j v_j^* - K_{j+1} v_j^* - Q'_j \underline{u}_j - Q''_j v_j^*] \\
&\Rightarrow Pd_j < \delta[K_{j+1} v_j^* + Q_j v_j^*] - \delta[K_{j+1} v_j^* + Q'_j \underline{u}_j + Q''_j v_j^*] \\
&\Rightarrow Pd_j + \delta[K_{j+1} v_j^* + Q'_j \underline{u}_j + Q''_j v_j^*] < \delta[K_{j+1} v_j^* + Q_j v_j^*]
\end{aligned}
$$

Thus (8) has been derived for $j = 1, 2, ..., N - 1$.

For player N

$$Pd_j < \delta[Q'_j v^*_j - Q'_j \underline{u}_j + Q''_j v^*_j - Q''_j v^*_j]$$
$$\Rightarrow Pd_j < \delta[(T-K+K_1)v^*_j - (T-K+K_1)v^*_j + Q'_j v^*_j + Q''_j v^*_j - Q'_j \underline{u}_j - Q''_j v^*_j]$$
$$\Rightarrow Pd_j < \delta[(T-K+K_1)v^*_j + Q_j v^*_j - (T-K+K_1)v^*_j - Q'_j \underline{u}_j - Q''_j v^*_j]$$
$$\Rightarrow Pd_j < \delta[(T-K+K_1)v^*_j + Q_j v^*_j] - \delta[(T-K+K_1)v^*_j + Q'_j \underline{u}_j + Q''_j v^*_j]$$
$$\Rightarrow Pd_j + \delta[(T-K+K_1)v^*_j + Q'_j \underline{u}_j + Q''_j v^*_j] < \delta[(T-K+K_1)v^*_j + Q_j v^*_j]$$

So the constraint for agent N has been derived too.

F Derivation of (1) from bound A*

Bound A* implies that

$$P' > \frac{(N-1)d}{(Np-1)\delta(v^*_i - \underline{u}_i)}$$
$$\Rightarrow P' > \frac{d}{\delta(v^*_i - \underline{u}_i)} \quad \text{as } \frac{N-1}{Np-1} > 1 \text{ as } p \in (\tfrac{1}{N}, 1]$$

Thus bound A has been derived. It has been shown above that (1) can be derived from bound A.

G Derivation of (11)

Bound A* and bound C taken together imply

$$Q'_i > PP' > P\left(\frac{(N-1)d}{(Np-1)\delta(v^*_i - \underline{u}_i)}\right)$$
$$\Rightarrow Q'_i > P\left(\frac{d(N-1)}{(Np-1)\delta(v^*_i - \underline{u}_i)}\right)$$
$$\Rightarrow \delta(v^*_i - \underline{u}_i)Q'_i > P\left(\frac{d(N-1)}{(Np-1)}\right) \quad \text{as } \delta(v^*_i - \underline{u}_i) > 0$$
$$\Rightarrow Q'_i \frac{(Np-1)}{(N-1)}\delta(v^*_i - \underline{u}_i) > Pd \quad \text{as bound G implies } \frac{Np-1}{N-1} > 0$$
$$\Rightarrow Q'_i \frac{(Np-1)}{(N-1)}\delta(v^*_i - \underline{u}_i) > Pd \geq Pd_i$$
$$\Rightarrow Q'_i \frac{(Np-1)}{(N-1)}\delta(v^*_i - \underline{u}_i) > Pd_i$$
$$\Rightarrow Q'_i \frac{(Np-1)}{(N-1)}(v^*_i - \underline{u}_i) > Q'_i \frac{(Np-1)}{(N-1)}\delta(v^*_i - \underline{u}_i) > Pd_i \quad \text{as } \delta \in (0,1)$$
$$\Rightarrow Q'_i \frac{(Np-1)}{(N-1)}(v^*_i - \underline{u}_i) > Pd_i$$

Thus (12) has been derived.

$$\Rightarrow Pd_i < \frac{Np-1}{N-1}Q'_i(v^*_i - \underline{u}_i)$$
$$\Rightarrow Pd_i < \frac{Np-1}{N-1}(Q'_i v^*_i - Q'_i \underline{u}_i)$$
$$\Rightarrow Pd_i < \frac{Np-1}{N-1}Q_i v^*_i - \frac{Np-1}{N-1}[Q'_i \underline{u}_i + Q''_i v^*_i]$$
$$\Rightarrow Pd_i + \frac{Np-1}{N-1}[Q'_i \underline{u}_i + Q''_i v^*_i] < \frac{Np-1}{N-1}Q_i v^*_i$$
$$\Rightarrow Pd_i + \frac{(N-1)p-1+p}{N-1}[Q'_i \underline{u}_i + Q''_i v^*_i] < \frac{(N-1)p-1+p}{N-1}Q_i v^*_i$$
$$\Rightarrow Pd_i + p[Q'_i \underline{u}_i + Q''_i v^*_i] - \frac{1-p}{N-1}[Q'_i \underline{u}_i + Q''_i v^*_i] < pQ_i v^*_i - \frac{(1-p)}{N-1}Q_i v^*_i$$
$$\Rightarrow Pd_i + p[Q'_i \underline{u}_i + Q''_i v^*_i] + \frac{(1-p)}{N-1}Q_i v^*_i + \frac{(1-p)}{N-1}\sum_{b=1, b\neq i, b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w^b_i] <$$
$$pQ_i v^*_i + \frac{(1-p)}{N-1}[Q'_i \underline{u}_i + Q''_i v^*_i] + \frac{(1-p)}{N-1}\sum_{b=1, b\neq i, b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w^b_i]$$

Thus (11) has been derived.

H Derivation of (13)

This is the same as the derivation for (11) except we do not remove the δ.

Bound A* and bound C taken together imply

$$
\begin{aligned}
&Q'_i > PP' > P(\tfrac{(N-1)d}{(Np-1)\delta(v_i^* - \underline{u}_i)}) \\
&\Rightarrow Q'_i > P(\tfrac{d(N-1)}{(Np-1)\delta(v_i^* - \underline{u}_i)}) \\
&\Rightarrow \delta(v_i^* - \underline{u}_i)Q'_i > P(\tfrac{d(N-1)}{(Np-1)}) \quad \text{as } \delta(v_i^* - \underline{u}_i) > 0 \\
&\Rightarrow Q'_i \tfrac{(Np-1)}{(N-1)}\delta(v_i^* - \underline{u}_i) > Pd \quad \text{as bound G implies } \tfrac{Np-1}{N-1} > 0 \\
&\Rightarrow Q'_i \tfrac{(Np-1)}{(N-1)}\delta(v_i^* - \underline{u}_i) > Pd \geq Pd_i \\
&\Rightarrow Q'_i \tfrac{(Np-1)}{(N-1)}\delta(v_i^* - \underline{u}_i) > Pd_i
\end{aligned}
$$

Thus (14) has been derived.

$$
\begin{aligned}
&\Rightarrow Pd_i < \delta\tfrac{Np-1}{N-1}Q'_i(v_i^* - \underline{u}_i) \\
&\Rightarrow Pd_i < \delta\tfrac{Np-1}{N-1}(Q'_i v_i^* - Q'_i\underline{u}_i) \\
&\Rightarrow Pd_i < \delta\tfrac{Np-1}{N-1}Q_i v_i^* - \delta\tfrac{Np-1}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] \\
&\Rightarrow Pd_i + \delta\tfrac{Np-1}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] < \delta\tfrac{Np-1}{N-1}Q_i v_i^* \\
&\Rightarrow Pd_i + \delta\tfrac{(N-1)p-1+p}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] < \delta\tfrac{(N-1)p-1+p}{N-1}Q_i v_i^* \\
&\Rightarrow Pd_i + \delta p[Q'_i\underline{u}_i + Q''_i v_i^*] - \delta\tfrac{1-p}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] < \delta p Q_i v_i^* - \delta\tfrac{(1-p)}{N-1}Q_i v_i^* \\
&\Rightarrow Pd_i + \delta p[Q'_i\underline{u}_i + Q''_i v_i^*] + \delta\tfrac{(1-p)}{N-1}Q_i v_i^* + \delta\tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b] < \\
&\delta p Q_i v_i^* + \delta\tfrac{(1-p)}{N-1}[Q'_i\underline{u}_i + Q''_i v_i^*] + \delta\tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq i,b\neq j}^{N}[Q'_b u_i(\underline{\alpha}^b) + Q''_b w_i^b]
\end{aligned}
$$

Thus (13) has been derived.

I Derivation of (15) from (16)

Recall equation (16) is identical to equation (10). Further recall that (10) was derived from bound D during the derivation of (4). Equation (16) therefore holds by bound D and states

$$\begin{aligned}
&\delta Q_i v_j^* = [\delta(Q_i' u_j(\underline{\alpha}^i) + Q_i'' w_j^i) - c_j^i] \\
\Rightarrow\ &\tfrac{Np-1}{N-1}\delta Q_i v_j^* = \tfrac{Np-1}{N-1}[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &\tfrac{(N-1)p}{N-1}\delta Q_i v_j^* - \tfrac{(1-p)}{N-1}\delta Q_i v_j^* = \tfrac{(N-1)p}{N-1}[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i] \\
&-\tfrac{(1-p)}{N-1}[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &p\delta Q_i v_j^* + \tfrac{(1-p)}{N-1}[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i] = p[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i] + \tfrac{(1-p)}{N-1}\delta Q_i v_j^* \\
\Rightarrow\ &p\delta Q_i v_j^* + \tfrac{(1-p)}{N-1}[\delta(Q_i' u_j(\underline{a}^i) + Q_i'' w_j^i) - c_j^i] + \tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq j,b\neq i}^{N}(\delta(Q_b' u_j(\underline{\alpha}^b) + Q_b'' w_i^b) - c_j^b) \\
&= p[\delta(Q_i' u_j(\underline{\alpha}^i) + Q_i'' w_j^i) - c_j^i] + \tfrac{(1-p)}{N-1}\delta Q_i v_j^* \\
&+\tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq j,b\neq i}^{N}(\delta(Q_b' u_j(\underline{\alpha}^b) + Q_b'' w_i^b) - c_j^b)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &p\delta Q_i v_j^* + \tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq j}^{N}(\delta(Q_b' u_j(\underline{\alpha}^b) + Q_b'' w_i^b) - c_j^b) \\
&= p[\delta(Q_i' u_j(\underline{\alpha}^i) + Q_i'' w_j^i) - c_j^i] + \tfrac{(1-p)}{N-1}\delta Q_i v_j^* \\
&+\tfrac{(1-p)}{N-1}\textstyle\sum_{b=1,b\neq j,b\neq i}^{N}(\delta(Q_b' u_j(\underline{\alpha}^b) + Q_b'' w_i^b) - c_j^b)
\end{aligned}$$

Thus (15) has been derived.

J Derivation of (17)

Bound H states that

$$\begin{aligned}
&T > \tfrac{NQd - c_i^j}{q}\sqrt{N} \\
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > \tfrac{NQ}{T}d - \tfrac{c_i^j}{T} \quad \text{as } N > 0 \text{ and } T > 0
\end{aligned}$$

Note that $d \geq v_i^* - n_i$ by definition. Therefore

$$\begin{aligned}
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > \tfrac{NQ}{T}(v_i^* - n_i) - \tfrac{c_i^j}{T} \\
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > \tfrac{(NQ)v_i^* - (NQ)n_i + Tv_i^* - Tv_i^* - c_i^j}{T} \\
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > \tfrac{Tv_i^* - (T-(NQ))v_i^* - (NQ)n_i - c_i^j}{T} \\
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > \tfrac{Tv_i^*}{T} - \tfrac{(T-(NQ))v_i^* + (NQ)n_i - c_i^j}{T}
\end{aligned}$$

Note that $\frac{(T-(NQ))v_i^* + (NQ)n_i - c_i^j}{T} = v_i$. This means that

$$\begin{aligned}
\Rightarrow\ &\tfrac{q}{\sqrt{N}} > v_i^* - v_i \\
\Rightarrow\ &\tfrac{q^2}{N} > (v_i^* - v_i)^2 \\
\Rightarrow\ &\textstyle\sum_{i=1}^{N}\tfrac{q^2}{N} > \sum_{i=1}^{N}(v_i^* - v_i)^2 \\
\Rightarrow\ &N \cdot \tfrac{q^2}{N} > \textstyle\sum_{i=1}^{N}(v_i^* - v_i)^2 \\
\Rightarrow\ &\sqrt{q^2} > \sqrt{\textstyle\sum_{i=1}^{N}(v_i^* - v_i)^2} \\
\Rightarrow\ &q > \|v^* - v\|
\end{aligned}$$

Thus (17) has been derived.

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