Asset Valuation
Debt Investments: Analysis and Valuation

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Study Session # 15 – Level I

CFA CANDIDATE READINGS:

Fixed Income Analysis for the Chartered Financial Analyst Program: Level I and II Readings, Frank J. Fabozzi (Frank J. Fabozzi Associates, 2000)
"Introduction to the Valuation of Fixed Income Securities," Ch. 5
"Yield Measures, Spot Rates, and Forward Rates," Ch. 6
"Introduction to Measurement of Interest Rate Risk," Ch. 7

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Learning Outcome Statements

Introduction to the Valuation of Fixed Income Securities
Chapter 5, Fabozzi

The candidate should be able to

a) *Describe the fundamental principles of bond valuation;*
b) *Explain the three steps in the valuation process;*
c) *Explain what is meant by a bond's cash flow;*
d) *Discuss the difficulties of estimating the expected cash flows for some types of bonds and identify the bonds for which estimating the expected cash flows is difficult;*
e) *Compute the value of a bond, given the expected cash flows and the appropriate discount rates;*
f) *Explain how the value of a bond changes if the discount rate increases or decreases and compute the change in value that is attributable to the rate change;*
g) *Explain how the price of a bond changes as the bond approaches its maturity date and compute the change in value that is attributable to the passage of time;*
h) *Compute the value of a zero-coupon bond;*
i) *Compute the dirty price of a bond, accrued interest, and clean price of a bond that is between coupon payments;*
j) *Explain the deficiency of the traditional approach to valuation in which each cash flow is discounted at the same discount rate;*
k) *Explain the arbitrage-free valuation approach and the role of Treasury spot rates in that approach;*
l) *Explain how the process of stripping and reconstitution forces the price of a bond toward its arbitrage-free value so that no arbitrage profit is possible;*
m) *Explain how a dealer can generate an arbitrage profit, and compute the arbitrage profit if the market price of a bond differs from its arbitrage-free value;*
n) *Explain the basic features common to valuation models that can be used to value bonds with embedded options.*

Yield Measures, Spot Rates, and Forward Rates
Chapter 6, Fabozzi

a) *Explain the sources of return from investing in a bond (coupon interest payments, capital gain/loss, and reinvestment income).*
b) *Compute the traditional yield measures for fixed-rate bonds (current yield, yield to maturity, yield to first call, yield to first par call date, yield to put, yield to worst, and cash flow yield);*
c) *Explain the assumptions underlying traditional yield measures and the limitations of the traditional yield measures.*
d) *Explain the importance of reinvestment income in generating the yield computed at the time of purchase.*
e) *Discuss the factors that affect reinvestment risks.*

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f) Calculate the reinvestment income that will be needed to generate the yield computed at the time of purchase.
g) Compute the bond equivalent yield of an annual-pay bond and compute the annual-pay yield of a semiannual-pay bond.
h) Calculate the discount margin measure for a floater and explain the limitation of this measure.
i) Compute, using the method of bootstrapping, the theoretical Treasury spot rate curve, given the Treasury par yield curve.
j) Compute the value of a bond using spot rates.
k) Explain the limitations of the nominal spread.
l) Describe the zero-volatility spread and explain why it is superior to the nominal spread.
m) Explain how to compute the zero-volatility spread, given a spot rate curve.
n) Explain why the zero-volatility spread will diverge from the nominal spread.
o) Explain the option-adjusted spread for a bond with an embedded option and what is meant by the option cost.
p) Illustrate why the nominal spread hides the option risk for bonds with embedded options.
q) Explain a forward rate.
r) Explain and illustrate the relationship between short-term forward rates and spot rates.
s) Compute spot rates from forward rates and forward rates from spot rates.

Introduction to the Measurement of Interest Rate Risk
Chapter 7, Fabozzi

a) Distinguish between the full valuation approach and the duration/convexity approach for measuring interest rate risk, and explain the advantage of using the full valuation approach.
b) Compute the interest rate risk exposure of a bond position or of a bond portfolio, given a change in interest rates.
c) Explain why it is difficult to apply the full valuation approach to a bond portfolio with a large number of positions, especially if the portfolio includes bonds with embedded options.
d) Explain and illustrate the price volatility characteristics for option-free bonds when interest rates change (including the concept of "positive convexity").
e) Explain and illustrate the price volatility characteristics of callable bonds and prepayable securities when interest rates change (including the concept of "negative convexity").
f) Compute the duration of a bond, given information about how the bond's price will increase and decrease for a given change in interest rates.

g) Compute the approximate percentage price change for a bond, given the bond's duration and a specified change in yield.
h) Explain, using both words and a graph of the relationship between price and yield for an option-free bond, why duration does an effective job of estimating price changes for small changes in interest rates but is not as effective for a large change in rates.
i) Distinguish between modified duration and effective (or option-adjusted) duration.
j) Explain why effective duration, rather than modified duration, should be used for bonds with embedded options.
k) Explain the relationship between modified duration and Macaulay duration and the limitations of using either duration measure for measuring the interest rate risk for bonds with embedded options.
l) Describe the various ways that duration has been interpreted and why duration is best interpreted as a measure of a bond or portfolio's sensitivity to changes in interest rates.
m) Compute the convexity of a bond, given information about how the price will increase and decrease for a given change in interest rates;
n) Compute the estimate of a bond's percentage price change, given the bond's duration and convexity and a specified change in interest rates.
o) Explain the difference between modified convexity and effective convexity.
p) Explain the importance of yield volatility in measuring the exposure of a bond position to interest rate risk.

"Discounted Cash Flow Applications"

a) Calculate the bank discount yield, holding period yield, effective annual yield, and money market yield for a U.S. Treasury bill;
b) Convert among holding period yields, money market yields, and equivalent annual yields;
c) Calculate the price and yield to maturity of a zero-coupon bond;
d) Explain the relationship between zero-coupon bonds and spot interest rates;

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Introduction to the Valuation of Fixed Income Securities

Chapter 5, Fabozzi

Introduction:

Chapter 5 provides a discussion of bond valuation, price-yield relationships and other valuation concepts related to option-free bonds. This is a chapter that is rich in short computational problems. Level I candidates need to be careful to cover all of the many LOS, but not delve too deeply in some esoteric areas that may distract from the overall objective.

Valuation Principles:

Describe the fundamental principles of bond valuation

The **valuation** process involves several steps in determining the fair value of the financial asset. Although the valuation can be complicated by a series of embedded options, this chapter will focus on "option-free" bonds in which the valuation is primarily determined through a simple process of discounting future *interest and principal payments*.

Explain the three steps in the valuation process

1. Estimate the cash flows (e.g. interest and principal payments)
2. Determine the appropriate discount rate
3. Calculate the present value of the cash flows

Explain what is meant by a bond's cash flow

The **cash flows** of the bond include both the principal and interest payments. Periodically, the interest payments change if they are based on a floating rate basis. Also, the principal payments may be changed based on unanticipated prepayments, calls or

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refunding situations. Coupon payments are assumed to accrue on a semiannual basis unless otherwise notified. Also, if there is no prior notification, bonds are assumed to mature at par value (typically $1000). Thus, if an 8% bond ($40 coupon payments every six months) matures in 1.5 years, the cash flows will appear as follows:

Year	.5	1	1.5
Coupon	$40	$40	$40
Principal			$1,000

In this case, the $40 payments represent the semiannual coupon amount and the $1000 represents the principal value repaid at par. In order to compute a bond value, the cash flows corresponding with the bond (e.g. semiannual coupon payments and principal) would be discounted at the yield to maturity (YTM) or investor's required return (this would be the "market" return or return demanded for this type of investment) for this bond corresponding with the period of receipt. The following sections explore this area in greater detail.

Discuss the difficulties of estimating the expected cash flows for some types of bonds and identify the bonds for which estimating the expected cash flows is difficult

There are several factors that cause difficulties in estimating bond cash flows:

1. The issuer has the option to change the contractual due date of the repayment of principal. This includes: callable bonds, putable bonds, mortgage-backed securities and asset-backed securities.
2. The coupon payment is reset based on a formula dependent on reference rates, prices or exchange rates. This includes floating-rate securities.
3. The investor has the option to convert or exchange the security into common stock. This includes convertible bonds and exchangeable bonds.

Compute the value of a bond, given the expected cash flows and the appropriate discount rates

A review of time value of money, bond pricing and return measures:

- Future values grow at the compounded interest rate
- Present values equal the future value discounted at the opportunity cost of capital.
- Lump sum values are distinguished from annuities.
- Annuities represent a constant flow of capital over a period of time and lump sums represent a single payment.
- Bond coupon payments represent annuity payments and the principal repayment is treated as a lump sum.

The present value of a future lump sum can be summarized by the following equation

$$PV = FV\left[\frac{1}{(1+i)^n}\right]$$

where,
PV = the present value
FV = the future value
I = the opportunity cost or discount rate
N = the number of periods

- The calculation of the bond value is simply adding the present value of the annuity payments to the present value of the lump sum principal repayment

- The value of the bond will increase with increases in the annuity payment (coupon) or the number of periods.
- For a given discount rate, the farther into the future a cash flow is received, the lower its present value.
- As the discount or implicit interest rate (opportunity cost of funds) decreases, the value of the bond increases (the reverse is also true).
- Decreases in the bond coupon or maturity will reduce the value of the bond just as an increase in the discount rate will reduce the value.
- Assume that the bond interest payment is semi-annual interest unless informed otherwise. (Note: for calculation purposes the number of payments are doubled and the market interest rate should be reduced by one half).

  - Yield (Internal Rate of Return): The internal rate of return (IRR) occurs when, present value of the cash flows = price of the investment.

  - IRR assumes cash payments are reinvested at the IRR rate.

  - Investors want investments where IRR > COC (cost of capital), ignores others

  - Semiannual rate can be "annualized" by multiplying the rate by 2.

When the number of periods increase, a more precise calculation may be required to account for the interest on interest.

Effective annual yield = (1 + Periodic interest rate)^{m} - 1

where, m = frequency of payments per year

Example, 4% compounded semiannually equals annualized yield:
8.16% = $(1.04)^2$ - 1 = 1.0816 - 1 = 8.16%

Bond Pricing:

- Bond prices change with changes in interest rates.
- Interest rates change with Fed policy, inflation or risk perception.
- Bonds that sell at a discount or premium accrete toward par as they mature
- Coupon payments never change, unless the bonds are a variable rate interest.
- Interest payments remain constant throughout the life of the bond.
- Changes in risk perception or discount rate cause the bond to rise or fall.
- Bonds rarely are sold on the date of the coupon payment (this only happens twice per year).
- Majority of bonds are sold between coupon payment dates (accrued interest).
- Buyers pay accrued interest on Treasuries (365 day year), and corporate, municipal and agency bonds on a "presumed" 30 day- month/360 day year.

Example:

Compute the value of a bond assuming a 6% semiannual coupon, 3 years to maturity and market interest rates of 9%.

Solution:

Year	.5	1	1.5	2.0	2.5	30
Coupon	$30	$30	$30	$30	$30	$30
Principal						$1,000

In this case, the $30 payments represent the semiannual coupon amount (e.g. $60/2) and the $1000 represents the principal value repaid at par. In order to compute a bond value, the cash flows corresponding with the bond need to be discounted at the market's semiannual discount rate. Since the market interest rate is 9%, each cash flow will be discounted at 4.5% (e.g. 9%/2). The final calculation is easily handled with a financial calculator.

Computation steps using an HP 12C:
PMT = 30 (semiannual coupon)
I = 4.5% (semiannual market interest rate)
FV = 1000 (principal value at par)
N = 6 (number of semiannual periods—3 years *2)
PV = solve = $922.63

Explain how the value of a bond changes if the discount rate increases or decreases and compute the change in value that is attributable to the rate change

The value for a 6%, 3 year bond given market interest rates of 9% equals $922.63. However as the market interest rates increase, the value of the bond declines (and vice versa). The table below demonstrates how this 6% coupon, 3 year bond changes in value from $853 to $1085, by dropping the market discount rate from 12% to 3%.

Coupon	Maturity (term)	Market Interest Rate	Bond Value
6%	3 years	9%	$922.63
6%	3 years	12%	$852.48
6%	3 years	6%	$1,000.00
6%	3 years	3%	$1,085.46

Solution:
PMT = 30 (60/2 reflecting semiannual payments)
N = 6 (3 years * 2 compounded semiannually)
FV = 1000
I = 6 (semiannual rate for 12% market rate)
Solve for PV = $852.48

I = 3 (semiannual rate for 6%)
Solve for PV = $1000 (note: no calculation should be required—market interest rates equal coupon payments so the bond should trade at par or $1000)

I = 1.5 (semiannual rate for 3%)
Solve for PV = $1,085.46

The relationship between a bond's coupon rate, the yield required by the market, and the bond's price relative to par value (i.e. discount, premium, or par value) are often important areas on the Level I exam. The following chart will assist with these concepts.

Relationship between price, coupon, YTM and par value:

Bond selling at	**Relationship**				
Par	Coupon rate	=	current yield	=	YTM
Discount	Coupon rate	<	current yield	<	YTM
Premium	Coupon rate	>	current yield	>	YTM

Notice in the prior example, that the bond value equaled par ($1,000) when market interest rates were 6% and matched the coupon rate. Alternatively, when the market interest rates rose to 12%, or a level well above the coupon rate of 6%, the bond value dropped below par to an amount of $852. Finally, when market interest rates dropped to 3%, or below the coupon, the bond increased in value to $1,085. Candidates that can quickly recognize the relationships between the coupon, par value and bond value, might be able to save valuable time on the AIMR CFA examination, by eliminating several answer selections on the exam without any computation.

For example, suppose AIMR asked a question such as:

Question:
What is the bond value for a 6%, 3 year bond when market interest rates are 12%?
A $1,000
B $1,085
C $1,195
D $ 852

Solution (Non quantitative):

In this situation, the Level I candidate should be able to quickly select answer "D" without calculation, since it is the only bond in the group that is trading below par! Candidates need to be efficient in test taking so that time can be allocated to other areas that might really need detailed computations.

Explain how the price of a bond changes as the bond approaches its maturity date and compute the change in value that is attributable to the passage of time

As a bond approaches maturity it has less variability with changes in market interest rates. For example, taken to the extreme, bonds prices will change very little the day or two before maturity. However, as bond maturities increase, there will be greater change in the bond price given a change in market interest rates. Note that the price increase or decrease will depend on the bond coupon, market interest rate and maturity. Candidates should also recognize that as the coupon increases the value of the bond increases (and vice versa) as shown in the example below.

A summary of bond maturity and value is as follows:

1. A bond selling at a premium decreases in value over time as the bond approaches maturity.
2. A bond selling at a discount increases in value over time if the bond is selling at a discount.
3. A bond selling at par remains unchanged as it approaches maturity.

The value for a 6%, 10-year bond given market interest rates of 9% equals $804.88. However as the bond approaches maturity, the value of the bond increases (and vice versa). The table below demonstrates how a 6% coupon, 10- year bond changes in value from $805 to $972, by reducing the term to maturity from 10 years to 1 year. At maturity the bond will sell for par value (assumed here to be $1,000).

Coupon	**Maturity (term)**	**Market Interest Rate**	**Bond Value**
6%	1 year	9%	$971.91
6%	4 years	9%	$901.06
6%	7 years	9%	$846.66
6%	10 years	9%	$804.88

Solution for 10-year maturity:
PMT = 30 (60/2 reflecting semiannual payments)
N = 20 (10 years * 2 compounded semiannually)
FV = 1000
I = 4.5 (semiannual rate for 9% market rate)
Solve for PV = $804.88

Solutions for 7, 4 and 1-year maturities only require N to change:

Solution for 7-year maturity,

N = 14;
Solve for PV = $846.66

Solution for 4-year maturity,
N = 8
Solve for PV = $901.06

Solution for 1-year maturity,
N = 2
Solve for PV = $971.91

Compute the value of a zero-coupon bond

The value of a zero coupon bond represents a single payment at the date of maturity. Consequently, the value of the zero coupon bond decreases as the maturity increases (and the reverse holds true). The example below shows how a zero coupon bond with a 3-year maturity, receives no annual payments and only one final payment at the end of the term. The value of this zero coupon bond, assuming a market interest rate of 8%, is $790.31. It is important to note that the convention for computing the zero coupon bond value assumes semiannual compounding on the principal value. Thus, the number of periods is doubled (just as with a conventional bond) and the market interest rate is divided in half (same as conventional bond). The solution is shown below:

FV = 1000
N = 6 (3 years *2=6; assumes semiannual compounding)
I = 4 (8%/2 = 4)
PMT = 0 (note: when using the HP 12C it is not necessary to place any value for PMT)
Solve for PV = $790.31

Year	.5	1	1.5	2.0	2.5	3.0
Coupon	$0	$0	$0	$0	$0	$0
Principal						$1,000

The value for a zero coupon 3-year bond, given market interest rates of 8%, equals $790. However as the bond maturity increases, the value of the bond decreases (and vice versa). The table below demonstrates how a zero coupon bond changes in value from $790 to $95, by increasing the term to maturity from 3 years to 30 years. These long-term to maturity, low (or zero) coupon bonds are commonly referred to as “deep discount" bonds. Deep discount bonds are extremely sensitive to market interest rate changes owing to the dependence on the final maturity payment. At maturity zero coupon or deep discount bonds will sell for par value (assumed here to be $1,000).

Coupon	**Maturity (term)**	**Market Interest Rate**	**Bond Value**
0%	3 years	8%	$790.31
0%	10 years	8%	$456.39
0%	20 years	8%	$208.29
0%	30 years	8%	$95.06

Solution for 10-year maturity
FV = 1000
N = 20 (10 * 2=20)
I = 4 (8%/2 = 4)
PMT = 0 (note: when using the HP 12C it is not necessary to place any value for PMT)
Solve for PV = $456.39

Solution for 20-year maturity (note: the computations for different maturities only require N to change)
N = 40 (20 *2); N = 60 for 30 year bond (30 * 2)
Solve for PV = $208.29 for 20-year zero coupon; Solve for PV = $95.06 for 30-year zero coupon bond

Compute the dirty price of a bond, accrued interest, and clean price of a bond that is between coupon payments

The price of the bond including accrued interest is known as the **full price** or **dirty price**. The **price** of the bond or **clean price** is equal to the full price less the accrued interest. The clean price is equal to the discounted cash flows shown in the examples above.

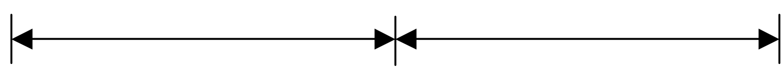
The clean price becomes more complicated when the calculation requires a partial period. Investors with access to a Bloomberg terminal or calculator enabling a partial period calculation can easily estimate the clean price of a bond. The data is simply entered in along with the beginning and ending dates. However, many calculators do *not* allow for partial periods. Since AIMR requires Level I candidates to calculate dirty and clean prices, the following discussion illustrates the desired approach (assuming a calculator enabling a partial period entry) as well as a convenient, alternative approach (for those without calculators enabling a partial period).

The amount that the buyer pays the seller encompasses two components related to the next cash flow. These include:

A. Interest earned by the seller
B. Interest earned by the buyer

The total interest paid to the new owner of the bond (i.e. buyer) represents the difference between the last coupon payment and the next coupon payment. The interest earned by the seller includes the accrued interest determined by the difference between the last coupon payment and the settlement date. This is shown in the exhibit below as section “A”. The interest earned by the buyer is the difference between the settlement date and the next coupon payment date. This is shown in the exhibit below as section “B”. The computation of the fractional period is determined below the exhibit.

“A” Interest accrued to seller | “B” Interest accrued to buyer



Last Coupon Date | Settlement Date | Next Coupon Date

Computation of Full price and partial periods:

The Full price is computed by determining all of the complete periods as well as the fractional period. Since the buyer is entitled to only the interest between the settlement date and the next coupon date (e.g. section “B”), the bond price represents all of the complete periods plus the partial period representing the percent allocation of section “B”. This ratio can be easily determined as:

$$\text{w periods} = \frac{\text{Section “B” (e.g. days between settlement period and next coupon date)}}{\text{Total days in the coupon period (e.g. Section A+ Section B)}}$$

Problem:
What is the clean price of a bond that has a 7% coupon and 10 remaining semiannual payments? Assume that the market interest rate is 5% and there are 58 days prior to the next coupon payment (assume 183 days in the coupon period).

Problem:
Compute the full price for the bond described above.

Solution:
Calculation for the **full price** (**dirty** price): If the next coupon payment is in 58 days, then the calculation for the next period should be $35/(1.025)^{58/183}$. The partial period of

time (w) is 58/183 (.317) of one period. Moreover, the second payment should be $\$35/(1.025)^{1+.317}$, the third payment should be $\$35/(1.025)^{2+.317}$… with the final payment being $\$1035/(1.025)^{9+.317}$. The calculation for the dirty price is equal to $1106.02.

Accrued Interest

Once the full (dirty price) is ascertained, the investor can subtract the accrued interest which is equal to $35 * (1-.317) = $23.91. The value .317 is subtracted since this equals the amount of interest owing to the new buyer. The ratio of 1 - .317 (1 – w) reflects the 125 days (183-58) that are attached to the bond as accrued interest. In effect the new buyer is purchasing a bond, which will pay a coupon of $35 in 58 days. Therefore, the bond has already earned $23.91 in interest that must be paid to the seller of the bond.

The amount of accrued interest depends on the number of days in the accrued interest period divided by the number of days in the coupon period. This ratio is then multiplied by the coupon payment to determine the accrued interest. The formula for accrued interest follows:

$$\text{Accrued interest} = \text{annual coupon}/2 * \frac{\text{Days in Accrued Interest period}}{\text{Days in coupon period}}$$

Clean Price

The full price includes this accrued interest ($1106.02) and the clean price equals the full price less the accrued interest ($1,106.02 - $23.91 =$1,082.11). Since many calculators do not enable a partial period (e.g. 1.317 periods, etc.), it will be difficult if not impossible for Level I candidates to calculate the full or clean price in this manner. However all is not lost. The following will provide a relatively simple estimate for the clean price calculation. Please recognize that this calculation is not included in your Fabozzi textbook.

Alternative approach to calculation

Estimating procedure for calculating a clean bond price. Assume the same facts as before. What is the clean price for a bond with a 7% coupon and 10 remaining payments? Assume that the next payment is in 58 days with 183 days in the period. Further assume that the market interest rate is 5%.

First, calculate the value with 10 periods:

n=10, pmt =35, fv =1000, I=2.5%--pv= $1087.52

Second, calculate the value with 9 periods:

n=9, pmt =35, fv =1000, I=2.5%--pv= $1079.71

Third, interpolate between the first value and second value based on number of days between first and second payments.

=$1,087.52 - $1,079.71 = 7.81 * 58/183 = 2.475 **[$1,079.71 + $2.48 =$1,082.19]**

The value of $1,082.19 is similar to the value above. The full price is then determined as before in the opposite manner. Take the $1,082.19 + accumulated interest [$35*(1-58/183) = $1,106].

This approach will be very helpful for candidates that do not have the ability to enter partial periods into their calculators.

Day count conventions:

The day count conventions vary depending on the security in question.

Actual/Actual*:* Treasury securities use the "actual/actual" day count convention. This approach counts the actual days between the settlement date and next coupon. This approach also uses the actual number of days in the coupon period. Note that the settlement date is not counted.

Example:
Assume a coupon-bearing Treasury security is sold with a settlement date of June 15th. Assume that the next coupon payment is September 1st. The number of actual days between June 15th and September 1st (the date of the next coupon payment) are 78 days as shown below:

June 15 to June 30	15 days
July	31 days
August	31 days
September 1	1 day
Total	78 days

The settlement date (June 15th) is not counted. Moreover, the number of actual days between March 1 and September 1 is 184 days. The number of days between the last coupon date (March 1) and June 15 is consequently 106 days (184 – 78).

30/360: Coupon-bearing agency securities, municipal bonds and corporate bonds use a day count convention known as a "30/360". This convention assumes that each month has 30 days; a six-month period has 180 days and a one-year period has 360 days.

Example:
Assume a coupon-bearing corporate security is sold with a settlement date of June 15th. Assume that the next coupon payment is September 1st. The number of days between June 15th and September 1st (the date of the next coupon payment) are 76 days as shown below:

June 15 to June 30	15 days

July	30 days
August	30 days
September 1	1 day
Total	76 days

Notice in the 30/360 convention that both July and August credit only 30 days interest rather than the actual 31 days in the calendar month. The settlement date is not counted in this calculation. The number of days counted in the accrued interest period is 104 days based on the 180-day period less the 76 days between settlement and next coupon.

Explain the deficiency of the traditional approach to valuation in which each cash flow is discounted at the same discount rate.

The traditional approach views each security as the same package of cash flows irrespective to term to maturity. It assumes that the investor is indifferent to higher or lower coupons as long as the implicit interest rate is the same on a comparable risk security. For example, three Treasury securities including an 11% coupon, 8% coupon and a zero coupon, each with a 10-year maturity priced to a 10% on-the-run security, would suggest that investors are indifferent to the timing of the cash flows (since the Treasury is considered default-free). However, if the three bonds were each viewed as holding a series of distinct zero coupon securities, they might be priced differently. The next section, which computes a Treasury value based on an arbitrage-free valuation approach, incorporates different discount rates corresponding to different spot rates prevailing at different terms to maturity. The YTM viewed in the market, is in reality, a weighted average composite of different spot rates used to discount the cash flows of the security.

Explain the arbitrage-free valuation approach and the role of Treasury spot rates in that approach.

The **arbitrage-free valuation approach** computes the value of a security assuming that it is converted into a series of zero-coupon bonds. The arbitrage-free approach is the theoretical value of a U.S. Treasury security if it issued a zero-coupon bond with a maturity date equal to the maturity date of each coupon and principal cash flow. This approach may provide a higher valuation than the traditional method. As mentioned previously, the term given for the zero-coupon Treasury rate is the **spot rate**. The **arbitrage-free value** is the value of a bond based on spot rates applied to each corresponding cash flow. The traditional approach discounts each cash flow at the same rate.

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Explain how the process of stripping and reconstitution forces the price of a bond toward its arbitrage-free value so that no arbitrage profit is possible.

In the absence of Treasury strips, a dealer could buy a Treasury security, strip the coupons, and bundle a package as a series of zero-coupon securities. However, since Treasury strips exist in the marketplace, market demands push the prices toward an arbitrage-free value. Thus, the elimination of arbitrage profits push values toward an "arbitrage-free value, or the value of the Treasury if each payment were discounted at the spot rate.

Assume the bond below has a 9% coupon and is priced at $900. Given the 2 year remaining term, the bond has a yield to maturity of 14.97% (n =4, PV = 900, pmt = 45, fv = 1000 solve for I (and double for annualized rate).

Traditional Approach

Year	.5	1	1.5	2.0
Coupon	$45	$45	$45	$45
Principal				$1,000

If Bond Price = $900 then YTM = 14.97%
Solution:
N = 4
PV = 900
FV = 1000
PMT = 45
Solve for I and double = 14.97%
Discounting all cash flows at an annualized rate of 14.97% provides a value = $900
Proof:

Year	.5	1	1.5	2.0
Coupon	$45	$45	$45	$45
Principal				$1,000

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PV $41.86 + $38.95 + $36.25 + $782.94 = $900
(Discounted at 14.97/2)

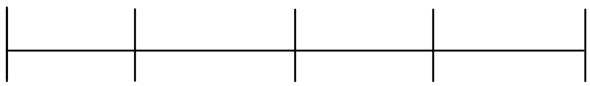
In the traditional approach, all cash flows are discounted at the YTM rate. Thus, in the example above, all cash flows are discounted at 14.97%. When all discounted cash flows are added together they equal a price of $900.

Arbitrage Approach:

Now assume that the spot rates for the 9%, 2-year bond in each period were 15%, 16%, 15% and 14%. The arbitrage-free valuation would be $913.89. The arbitrage free valuation presumes that each cash flow is discounted at the prevailing market interest rate for that specific term. Thus, the $45 cash flow for the first period (after six months) would be discounted at an annualized rate of 15% (7.5% semiannual rate) and would equal $41.86. The cash flows for the second period would be worth $38.58 and so forth. Adding all of the discounted cash flows together provides a value equal to $913.89.

Arbitrage-Free Approach

Year	.5	1	1.5	2.0



Coupon	$45	$45	$45	$45
Principal				$1,000
Spot rate	15%	16%	15%	14%

Discounted
Cash flow 41.86 + 38.58 + 36.22 + 797.23 = $913.89

Since a dealer can buy the security in the marketplace for $900 and strip the security and earn a value of $913.89 using Treasury spot rates, there is an opportunity for an arbitrageur's profit of $13.89 (913.89 – 900). Presumably such an opportunity, if it did exist, would last for only a short while, since this provides a risk-free mechanism for the dealer to earn a profit. If the bond sold for $925 rather than $900, the dealer would have an incentive to buy up the Treasury strips (for a value of $913.89) and repackage as a synthetic Treasury security for $925. This process, known as **reconstitution** provides the dealer with a profit of $11.11 (profit of reconstitution = $925 – $913.89 = $11.11).

Explain how a dealer can generate an arbitrage profit, and compute the arbitrage profit if the market price of a bond differs from its arbitrage-free value.

If an 8%, 10 year Treasury was valued at a market discount rate of 6%, the price of the security would be 114.88 assuming that each cash flow was discounted at the same annualized rate of 6%. However, assume that the spot rates for the individual cash flows (if viewed as a series of zero coupon securities) created a price of 115.27, then the dealer has incentive to buy the 10 Treasuries, strip the security and net an arbitrage profit equal to 115.27-114.88.

The opportunity to create an arbitrage profit will exist whenever the market value is less than the arbitrage-free value. If the Treasury's market price is greater than the arbitrage-free value, the dealer has no incentive to strip the security. In such cases the dealer can reverse the direction and create a synthetic Treasury by buying Treasury strips in a process known as **reconstitution**. Given the opportunity to easily strip a Treasury or create a synthetic Treasury security through reconstitution, U.S. Treasury securities tend to trade at or very close to their arbitrage-free prices. Arbitragers would either bid prices up or down until such arbitrage profits could no longer be earned.

Explain the basic features common to valuation models that can be used to value bonds with embedded options.

Generally, bonds have a **term structure of credit** spreads in which the credit spreads increase with the maturity of the bond. Moreover, the term structure is not the same for all credit ratings. Low credit ratings have a steeper term structure of credit spreads. When the credit spreads for the given credit rating and market sector are added to the Treasury spot rates a **benchmark spot rate curve** can be created. This spot rate curve enables an investor to discount the cash flows of a non-Treasury security based on higher credit risk and a non-linear term structure of interest rates (i.e. spread varies over time).

A **binomial model** is often used to value callable bonds, putable bonds, floating rate notes, and structured notes when the coupon is based on an interest rate. A **Monte Carlo simulation model** is often used to estimate the value of mortgage-backed securities and other asset-backed securities. There are five basic features of each valuation model. Each model begins with an on-the-run Treasury security and Treasury spot rate. Then the model incorporates an assumption about expected volatility and interest rate "paths". The model is then programmed to follow the Treasury along with assumptions regarding the conversion of the option (call, prepayment, etc.). In these cases the investor is exposed to **modeling risk** in which the assumptions may have been entered incorrectly.

Problems: Fabozzi, Chapter 5

1. What is the value of a bond with a 7% coupon, 5-year maturity and market interest rate of 12%?
   A. $816.00
   B. $936.22
   C. $1,011.46
   D. $1,230.97

2. What is the value of a bond with a 7% coupon, 5-year maturity assuming market interest rates at 6%?

   A. $827.09
   B. $987.16
   C. $1,237.88
   D. $1,042.65

3. What is the value of a bond with a 6% coupon, a 5-year maturity and market interest rate of 12%?
   A. $816.00
   B. $779.20
   C. $914.43
   D. $1,167.92

4. What is the value of the bond with a 6% coupon, 12 % market interest rate and a 10-year maturity?

   A. $856.81
   B. $779.20
   C. $655.90
   D. $1077.43

5. What is the value of a bond with a 0% coupon, 12 % market interest rate and a 10-year maturity?
   A. $1,103.82
   B. $856.81
   C. $655.90
   D. $311.80

6. On May 25th, Marcia Wilson purchased a 10%, 15-year Treasury bond, with semiannual interest payments occurring April 1 and October 1 each year. What is the accrued interest on this bond?

A. $35.25
B. $70.50
C. $29.50
D. $14.75

7. On May 15th, Arjun Ramswy purchased a 10%, 15-year bond corporate bond, with semiannual interest payments occurring April 1 and October 1 each year. What is the accrued interest on this bond?

   A. $12.22
   B. $37.78
   C. $24.44
   D. $75.56

8. Jan Lofblad purchased an 8%, 12-year corporate bond on November 20th for a full price of $1085.23. Her next coupon will be paid on March 1st. What is the clean price for this bond?

   A. $1,057.17
   B. $1,107.17
   C. $1,063.29
   D. $1,041.35

9. Rohan Smithers, analyst at Fundquest, purchased an 8% IBM bond with a settlement date of April 12, 2001. The bond matures June 1, 2005 and pays a semiannual interest. What is the fractional coupon period (i.e. the “w” value) with this bond?
   A. .27
   B. .73
   C. .23
   D. .77

10. Stephan Novackee, bought a 6% 10/01/11 corporate bond with a settlement date of February 20, 2001. Stephan’s bond pays semiannual interest on April 1 and October 1 each year. If the clean price for this bond is $933.22, what is the dirty price?
    A. 10.0
    B. 7.5
    C. 5.0
    D. Insufficient information to calculate

11. Michael Merley, a dealer for Smith, Barney, Rubble investment house, has observed a 2-year 10% coupon Treasury security selling for $900. Michael knows that the annualized spot rates for Treasury securities range from 14% to 16% (shown below).

Year	.5	1	1.5	2.0
$900 Market Price				
Coupon	$50	$50	$50	$50
Principal				$1,000
Spot rate	15%	16%	15%	14%

How can Michael take advantage of this situation?

A. He should reconstitute the spot rate securities by creating a synthetic Treasury
B. He should buy the 2 year Treasury and strip and sell the coupons and principal at the spot rates
C. He should sell the 2-year Treasury and buy securities selling at the spot rate
D. He should buy and hold the 2-year Treasury security

Answers: Fabozzi, Chapter 5

1. **A**. The bond price can be computed by entering the data as follows: n=10 (5 years compounded semiannually or 5 * 2=10), pmt = 35 (70 coupon paid semiannually or 70/2=35), I=6% (market interest rate of 12/2 = 6%), fv = 1000 (par value at maturity, solve for PV =$816.
2. **D.** The bond price can be computed by entering the data as follows: n=10 (5 years compounded semiannually or 5 * 2 =10), pmt = 35 (70 coupon paid semiannually or 70/2 = 35), I=3% (market interest rate of 6%/2 = 3%), fv =1000, solve for PV = $1042.65. Notice that in problem 1 (above) the terms for the bond are identical to this problem except that the market interest rates above were at 12% (annual) versus a market interest rate of 6% in this case. When the market interest rates are above the coupon, the bond will sell for a discount as shown in problem 1. When market interest rates are below the coupon (as in this problem) the bond will sell for a premium.

3. **B.** The bond price can be computed by entering the data as follows: n=10 (5 years compounded semiannually or 5 * 2 =10), pmt = 30 (60 coupon paid semiannually or 60/2 = 30), I=6% (market interest rate of 12%/2 = 6%), fv =1000, solve for PV =$779.20. Notice that this problem is similar to the first problem except that the coupon is 6% rather than 7%. Since the coupon is now lower than before (e.g. 6% rather than 7%), alert candidates should realize that the price has to be lower than $816. Since answer ”B” is the only option that is lower than $816 (value of this bond with a 7% coupon), it must be the correct answer.

4. **C.** The bond price can be computed by entering the data as follows: n=20 (10 years compounded semiannually or 10 * 2 =20), pmt = 30 (60 coupon paid semiannually or 60/2 = 30), I=6% (market interest rate of 12%/2 = 6%), fv =1000, solve for PV =$665.90. Notice that this problem is similar to the last problem except that the term is 10 years rather than 5 years. Since the term to maturity is now longer than before, alert candidates should realize that the price has to be lower than $779.20. Thus, this question does not require any computations since answer ”C” is the only option that is lower than $779.20 (value of this bond with a 5-year term).

5. **D.** The bond price can be computed by entering the data as follows: n=20 (10 years compounded semiannually or 10 * 2 =20), pmt = 0 (this is a zero coupon bond, either “0” can be entered for PMT or PMT can be left blank), I=6% (market interest rate of 12%/2 = 6%), fv =1000, solve for PV =$311.80. Notice that this problem is similar to the last problem except that the coupon is now $0 rather than $30. Since the coupon is now longer a factor, alert candidates should realize that the price has to be lower than the last answer of $665.90. Thus, this question does not require any calculations since answer ”D” is the only option that is lower than $665.90 (value of this bond with a 6% coupon).

6. **D**. Days between settlement and next coupon period =

May 25 to May 31	6 days
June	30 days
July	31 days
August	31 days
September	30 days
October 1	1 day
Total	129 days

Since there are 183 days in the ½ year period between April 1 and October 1, there are 54 days in the accrued interest period (183 – 129).

Note: w periods = 129/183 = .705 Therefore, AI = semiannual coupon * (1 - .705) = 50 * .295 = $14.75

$$\text{Accrued interest} = \frac{\text{annual coupon}}{2} * \frac{\text{days in AI period}}{\text{days in coupon period}} = \frac{100}{2} * \frac{54}{183} = 14.75$$

7. **A**. Days between settlement and next coupon period =

May 15 to May 30	15 days
June	30 days
July	30 days
August	30 days
September	30 days
October 1	1 day
Total	136 days

Since there are 180 days in the corporate ½ year, there are 44 days (180 – 136) in the accrued interest period.

Note: w periods = 136/180 = .756 Therefore, AI = semiannual coupon * (1 - .756) = 50 * .244 = $12.22

$$\text{Accrued interest} = \frac{\text{annual coupon}}{2} * \frac{\text{days in AI period}}{\text{days in coupon period}} = \frac{100}{2} * \frac{44}{180} = 12.22$$

8. **C**. Days between settlement and next coupon period =

November 20 to November 30	10 days
December	30 days
January	30 days
February	30 days
March 1	1 day
Total	101 days

Since there are 180 days in the corporate ½ year, there are 79 days (180 – 101) in the accrued interest period. The accrued interest is $21.94. Note: w periods = 101/180 = .561
AI = semiannual coupon * (1 -.561) = 50 * .439 = $21.94

$$\text{Accrued interest} = \frac{\text{annual coupon}}{2} * \frac{\text{days in AI period}}{\text{days in coupon period}} = \frac{100}{2} * \frac{79}{180} = 21.94$$

Therefore the clean price, or the full price of the bond less accrued interest is equal to $1085.23 less $21.94 or $1,063.29.

9. **A**. Since the bond matures June 1, it must pay interest on this date and December 1 (six months apart). Therefore, the "w" represents the remaining period between the settlement date and the next coupon date divided by the total number of days in the payment period.

   Days between settlement and next coupon period =

April 12 to April 30	18 days
May	30 days
June	1 day
Total	49 days

   Since there are 180 days in the corporate ½ year, there are 131 days (180 – 49) in the accrued interest period.

   Calculation of the fractional "w" periods = 49/180 = .27

10. **C**. Days between settlement and next coupon period =

February 20 to end of February	10 days
March	30 days
April 1	1 day
Total	41 days

   Since there are 180 days in the corporate ½ year, there are 139 days (180 – 41) in the accrued interest period. The accrued interest is $11.58. Note: w periods = 41/180 = .23

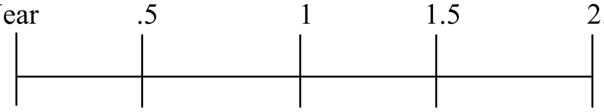
   Accrued Interest = semiannual coupon * (1 -w) = 15 * (1-.23) = $11.58

$$\text{Accrued interest} = \frac{\text{annual coupon}}{2} * \frac{\text{days in AI period}}{\text{days in coupon period}} = \frac{30}{2} * \frac{139}{180} = 11.58$$

Therefore the full price of the bond equals the clean price plus the accrued interest of \$11.58. Since the clean price is equal to \$933.22, the full price must equal \$933.22 + \$11.58 = \$944.80.

11. **B**. This question can be answered several ways. One approach is to calculate the implied YTM on the 2-year coupon security. The YTM can be calculated by:
N = 4
PV = -900
FV = 1000
PMT = 50
Solve for I = 8.04 * 2 (for semiannual interest) = 16.04%

Since the YTM is higher than any individual spot rate, the 2-year Treasury is priced low relative to the spot rates (i.e. higher yields). Thus, the analyst would want to buy the 2-year Treasury and sell at the lower yields (i.e. higher price). This can be accomplished by stripping the 2-year coupon and principal payments and selling at the spot rates. For proof of the arbitrage opportunity, you may discount each annual cash flow at the prevailing spot rate. This provides a discounted cash flow stream of \$930.67. Given the availability of the bond selling in the marketplace at \$900, an arbitrageur can purchase the bond at \$900, strip the coupons and principal for \$930.67 and earn a risk-free profit of \$30.67. In reality, such a large risk-free profit is unlikely given the number of market participants, however this provides the theoretical foundations for how an arbitrage profit situation operates.



Year	.5	1	1.5	2.0
\$900 Market Price				
Coupon	\$50	\$50	\$50	\$50
Principal				\$1,000
Spot rate	15%	16%	15%	14%

Discounted
Cash flow 46.51 + 42.87 + 40.25 + 801.04 = \$930.67

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Yield Measures, Spot Rates, and Forward Rates

Chapter 6, Fabozzi

Overview:

This chapter addresses a few fundamental areas regarding yield measures, spot rates and forward rates. Many of the computations discussed here regarding spot and forward rates, are also addressed in the Derivatives and/or Economics section. Since a number of short questions may be derived from this material, Level I candidates should be particularly focused on these overlapping sections.

Explain the sources of return from investing in a bond (coupon interest payments, capital gain/loss, and reinvestment income).

There are three primary sources of income from a bond.

1. **Coupon interest payments**—These amounts are set forth in the bond's indenture agreement and usually entail a fixed payment spread out over a discrete, previously defined period of time. Fixed payments are established as a percentage of par (i.e. 8%) and are usually paid on a semiannual basis. However, floating rate agreements may also be specified in the indenture, which provide a variable payment dependent upon a base rate and pre-specified index.
2. **Capital gain or loss** (associated with maturity, call or sale)—The difference between the purchase price and the maturity value (or sale price if sold prematurely) may yield a gain or loss (if different from original price). Significant gains or losses become more likely when the coupon rate varies considerably from the market interest rate.
3. **Reinvestment income**—This is the income that accrues from the coupon income being invested in an interest bearing account. Over time, the invested coupon payments and the accumulating interest-on-interest may become significant cash flows, and appreciably influence the overall required rate of return.

Compute the traditional yield measures for fixed-rate bonds (current yield, yield to maturity, yield to first call, yield to first par call date, yield to put, yield to worst, and cash flow yield).

Traditional yield measures:

Bond investors compare investments based on a variety of yield measures. This section introduces these yield measures as well as the steps necessary to perform the computations.

Yield Measure	Purpose
1. Nominal yield	Measures the coupon rate
2. Current yield	Measures current income rate
3. Bond equivalent Yield	Measures an annualized yield by doubling the semiannual return
4. Promised YTM	Measures expected rate of return for bond held to maturity
5. Promised YTC	Measures expected ROR for bond held to first call date
6. Realized/ Horizon	Measures expected ROR for bond likely to be sold prior to maturity date
6. Yield to Put	Measures expected ROR for bond held to first put date
7. Yield to Worst	Measures the worst possible yield given every possible call or put date
8. Yield on Treasury	Measures the discounted yield used for Treasuries
9. Cash flow yield	Measures the effective yield after incorporating prepayments of principal and actual interest

Yield to maturity (or promised yield to maturity). It considers specific reinvestment assumptions and estimated sales price. It can also measure the actual rate of return on a bond during some past period of time. The yield to maturity presumes that the coupon payments are reinvested at the bond's YTM rate. During periods of either very high or low interest rates this assumption may not hold. The mathematical formula is shown below, but in practice, the only important consideration is that this is the yield, which equates the discounted cash flows to the purchase price. It is determined on an iterative, or interpolating process, that performed manually, takes considerable time and effort. Although the formula is somewhat intimidating, the application and calculation couldn't be more straightforward. Fortunately, candidates need only remember that the YTM is the same as solving for "i" on a financial calculator. Either calculator that AIMR allows provides this computation by solving for the interest rate (e.g. "i").

Promised yield to maturity $P_m = \sum_{t=i}^{n} C_t \, 1/(1+i)^t$

Example: Kris Lynner bought an 8%, 5 year bond for $900. Compute the YTM of this bond.

Solution (using HP 12C)

PV = -900
N = 10 (5 * 2 assuming semiannual payments)
FV = 1000 (assume 1000 unless otherwise stated)
PMT = 40 (80/2 with semiannual payments)
I = solve for "I" is the same as YTM, this is the implied interest or yield earned on the bond given a price of $900 with an 8% coupon and 5 years to maturity.
I = 5.31 * 2 (interest needs to be annualized) = 10.63.

Nominal Yield--this is the coupon rate of a particular issue

Example: Kris Lynner bought an 8%, 5 year bond for $900. Compute the nominal yield for Kris's bond.

Solution: The nominal yield for this bond is the coupon of 8%.

Current Yield--the current yield is equal to: $CY = C_i / P_m$

where: CY = the current yield on a bond
C_i = the annual coupon payment of the bond
P_m = the current market price of the bond

Example: Kris Lynner bought an 8%, 5-year bond for $900. Compute the current yield for Kris's bond.

Solution: The current yield for this bond is the coupon of 8% divided by $900, or 80/900 = .0889 or 8.89%

Explain the assumptions underlying traditional yield measures and the limitations of the traditional yield measures.

The promised yield to maturity, otherwise known as YTM, is the most widely used bond yield calculation. It represents the fully compounded rate of return promised to the investor providing that two assumptions hold true:

Assumption #1: The investor holds the bond to maturity
Assumption #2: The investor reinvests all coupon interest payments back at the computed YTM rate (i.e. interest on interest is reinvested at the YTM rate)

Approximate promised yield (APY) or YTM (not included in Fabozzi Chapter 6)

$$APY = \frac{c_t + \frac{P_p - P_m}{n}}{\frac{P_p + P_m}{2}}$$

<u>Coupon + Annual Straight-Line Amortization of Capital Gain or Loss</u>
Average Investment

where:

P_p = par value of the bond
n = number of years to maturity
c_t = the bond's annual coupon
P_m = the current market price of the bond

PROMISED YIELD TO CALL

The promised yield to call (YTC) is computed the same as the promised YTM, except that the call price and call date are substituted for the maturity price and date whenever the investor deems that the likelihood of the bond being called is high. Bonds are typically subject to call whenever the YTC is less than the YTM, and the issuer has incentive to call the bond and finance the call with a new bond selling at (presumably) lower prevailing market interest rates.

$$AYC = \frac{c_t + \frac{P_c - P_m}{nc}}{\frac{P_c + P_m}{2}}$$

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P_c = call price of the bond
nc = number of years to first call date
C_t = the bond's annual coupon payment
P_m = the current market price of the bond

Example: Eric Richards bought an 8%, 5-year bond for $900. His bond has a call price of $1100. Assume market interest rates are expected to fall by 200 basis points in 3 years and the bond is called at its call price at that time. Compute the YTC of this bond.

Solution (using HP 12C)

PV = -900
N = 6 (3 * 2 assuming semiannual payments)
FV = 1100 (assume 1000 unless otherwise stated)
PMT = 40 (80/2 with semiannual payments)
I = solve for I is the same as YTM, this is the implied interest or yield earned on the bond given a price of $900 with an 8% coupon and 3 years to maturity.
I = 7.51 * 2 (interest needs to be annualized) = 15.02.

TOTAL DOLLAR RETURN OR REALIZED OR HORIZON YIELD

The realized yield or horizon yield incorporates the investor's expected rate of return on a bond, given that the bond is not held until maturity, or that the coupon interest payments are not reinvested at the bond's computed YTM. This yield incorporates all of the cash flows for the prevailing time period.

$$ARY = \frac{C_t + \frac{P_f - P_m}{hp}}{\frac{P_f + P_m}{2}}$$

ARY = approximate realized yield
P_f = future selling price of the bond
hp = holding period of the bond in years
C_t = the bond's annual coupon payment
P_m = the current market price of the bond

☑ *Other terms that Level I candidates should know*:

The **bond equivalent yield** is equal to double the semiannual yield. It is the annual measure used by many investors when applying the YTM.

Total future dollars equal all of the future cash flows that the investor will receive including: coupon payments, capital gain/loss and reinvestment income.

☑ **Interest rate risk** applies to a bond that is sold prior to maturity for a price that is below its purchase price. This occurs when interest rates rise.

Explain the importance of reinvestment income in generating the yield computed at the time of purchase.

Over a long period of time, the amount of interest may be influenced by the reinvestment rate as much as the actual coupon rate. Relatively low reinvestment rates influence the actual yield to be lower than what is implied in the YTM calculation. This happens because the YTM computation presumes that coupon payments are reinvested at the YTM rate. If an investor expects interest rates to fall, or if the investor believes that the reinvested coupon payments will be placed in a relatively low interest bearing account (i.e. checking account or savings account with a low interest rate), than the promised YTM will not be accomplished. In fact, the actual yield may be considerably lower than the promised YTM if the investment period is long and the reinvested return is low.

Example: Mike Jones purchased an 8%, 20 bond for $900. The promised YTM is 9.09%. Mike is planning on reinvesting the coupon payments in an interest bearing account yielding an annualized 3% rate. What is the actual yield that Mike will receive on his bond, assuming that, on-average he receives 3% additional interest on his coupon payments over the full life of his bond?

A. 9.09%
B. 12.09%
C. Greater than 9%
D. Less than 9%

Solution: Mike will receive well below 9% over the life of his bond. The promised YTM computation providing a return of 9.09% presumes that reinvested cash flows will generate 9.09%. Since the reinvested cash flows are, on average, providing only 3% the actual yield will be much lower than 9%.
This risk is known as the reinvestment risk and is particularly problematic for investors when interest rates decline.

☑ **Reinvestment risk** is the risk that an investor will reinvest future cash flows at a rate less than the YTM. This occurs when interest rates decline.

Discuss the factors that affect reinvestment risks.

Zero coupon bonds are the only bonds that have no reinvestment risk. They have no risk, since their value is not dependent on reinvested coupon payments. Other bonds, however have reinvestment risk to the extent that the reinvested cash flows (coupon and/or prepayments of principal) are reinvested at a rate that is different than the YTM rate. Increases in the bond's maturity and coupon increase the reinvestment risk.

- For a given YTM and a non-zero coupon, as the maturity increases the bond 's total dollar return becomes more dependent on reinvestment income (i.e. reinvestment risk increases with maturity).
- For a given maturity and YTM, as the coupon increases the bond's total dollar return becomes more dependent on the reinvestment of coupon payments (reinvestment increases with maturity).

Calculate the reinvestment income that will be needed to generate the yield computed at the time of purchase.

Suppose an investor purchases a 3-year, 8% coupon bond for a price of $949.24 that has a YTM equal to 10%. If the investor placed this money in a bank, it would accumulate to a total of $1,272.06 by the end of three years in order to generate a 10% rate of return. The composition of the $322.82 in cash flows $1,272.06 – 949.24) equals the total interest payments ($240) + the capital gain on the difference between the purchase price and final maturity value ($50.76) + the interest on interest accumulation ($32.06). The coupon payments and capital gain are easily computed. The coupon payments are simply the total number of periods (assuming semiannual payments) multiplied by the semiannual coupon. The capital gain or loss is simply the difference between the initial purchase price and the final maturity. The interest-on-interest computation, on the other hand, requires considerable effort. In order to compute this amount the analyst would need to compute the accumulated interest from receipt of payment until the end of the period (as shown below).

Interest payments (coupons)	= 40*6	=	$240
Capital gain	= 1000-949.24	=	$ 50.76
Interest on interest		=	$ 32.06
Total cash flow return		=	$322.82

The composition of the interest on interest ($32.06):

First $40 coupon reinvested for 5 periods at 5%	=$11.04
Second $40 coupon reinvested for 4 periods at 5%	=$8.62
Third $40 coupon reinvested for 3 periods at 5%	=$6.30
Fourth $40 coupon reinvested for 2 periods at 5%	=$4.10
Fifth $40 coupon reinvested for 1 periods at 5%	=$2.00
Sixth $40 coupon reinvested for 0 periods at 5%	=$0
Total interest on interest	= $32.06

Short-cut method:

While the likelihood of this question on a Level I exam is low, we show a convenient short-cut to this computation below. **Question:** Suppose an investor purchases a 3-year, 8% coupon bond for a price of $949.24 that has a YTM equal to 10%. Compute the reinvested interest on this bond.

Solution:
If the investor placed this money in a bank, it would accumulate to a total of $1,272.06 by the end of three years in order to generate a 10% rate of return [pv = $949.24, I= 5%, n = 6, solve for fv]. The composition of the $322.82 in cash flows $1,272.06 – 949.24) equals the $240 total interest payments ($40 * 6) + the $50.76 capital gain on the difference between the purchase price and final maturity value ($1000 – 949.24) + the interest on interest accumulation. Since the total cash flows equal $322.82, and the remaining two amounts are know, the balance must equal the interest-on-interest accumulation ($322.82 - $240 - $50.76 = reinvested interest). The $32.06 difference equals the reinvested interest accumulation.

Compute the bond equivalent yield of an annual-pay bond and compute the annual-pay yield of a semiannual-pay bond.

U.S. bonds normally pay semiannually whereas non-U.S. bonds, such as government bonds in Europe and Eurobonds tend to pay annually. To compute the bond equivalent yield of an annual bond an adjustment needs to be made.

$$\text{Bond-equivalent yield of an annual-pay bond} =$$

$$2[(1 + \text{yield on annual-pay bond})^{0.5} - 1]$$

The amount in the brackets provides the compounded semiannual yield equal to an annual –pay bond.
Doubling the semiannual yield provides the bond equivalent yield.

For example, the bond equivalent yield for an annual-pay bond with a YTM of 8% is:

$$2[(1 + .08)^{0.5} - 1] = 7.85\%$$

This demonstrates that the bond-equivalent yield will always be less than the annual-pay bond's YTM.

In order to convert a bond-equivalent yield of a U.S. bond to an annual pay basis, the following adjustment needs to be made:

$$\text{Yield on an annual pay basis} = [[(1 + \frac{\text{yield on bond-equivalent basis}}{2})^2 - 1]$$

The semiannual yield is first computed (i.e. bond-equivalent yield is divided by 2) and then the yield is compounded to arrive at a yield on an annual-pay basis.

For example, if a U.S. bond quoted a bond-equivalent yield of 7%, the YTM on an annual-pay basis would be equal to?

$$[(1 + .035)^2 - 1] = 7.12\%$$

Again, this shows that due to compounding, the yield on an annual-pay bond is always higher than the yield on a bond-equivalent basis.

Compute the yield to put

Some bonds come with a put option that enables the holder (buyer) to sell the bond back to the issuer prior to maturity. There are certain clauses that specify when (i.e. timing) such an action might occur (for example an event's provision) as well as the price. A put option might be exercised when interest rates have increased appreciably above the coupon rate causing the bond price to drop.

☑ **Yield to put** is the yield based on the first put date with the corresponding put price. There are two fundamental changes in computing this yield compared to the traditional YTM.

- Maturity date is shortened to reflect the earliest put date,
- Maturity value reflects the put price

The yield to put formula is otherwise identical to the computation for the yield to maturity or yield to call.

Example:
Luciana Kas purchased a 6%, 10-year bond for $850. The bond had its first put date at par, three years from now. Compute the yield to put on Luciana's bond.

Solution:
N= 6 (3*2—note: this is dependent on the number of periods to the first put date)
Pmt = 30 (60/2)
PV = -850
FV = 1000 (put price stated is $1000 or par)
Solve for I = 6.056 * 2 (annualized yield) = 12.11%

Compute the yield to worst

☑ **Yield to worst** is the lowest possible yield based on every possible call date and put date. In practice this yield has little meaning since it does not incorporate different potential yields based on reinvestment or interest rate risk.

Example:
Luciana Kas purchased a 6%, 10-year bond for $850. The bond had its first put date at par, three years from now and its first call date 5 years from now at $1100.
Compute the yield to worst on Luciana's bond.

Solution:

YTM
N= 20 (10*2—note: this is dependent on the number of periods to the maturity date)
Pmt = 30 (60/2)
PV = -850
FV = 1000 (assumed to mature at par)
Solve for I = 4.115 * 2 (annualized yield) = 8.23%

YTP (yield to put)
N= 6 (3*2—note: this is dependent on the number of periods to the first put date)
Pmt = 30 (60/2)
PV = -850
FV = 1000 (put price stated is $1000 or par)
Solve for I = 6.056 * 2 (annualized yield) = 12.11%

YTC (yield to call)
N= 10 (5*2—note: this is dependent on the number of periods to the first call date)
Pmt = 30 (60/2)
PV = -850
FV = 1100 (call price stated is $1100)
Solve for I = 5.78 * 2 (annualized yield) = 11.57%

Since the yields for the YTM, YTP and YTC are 8.23%, 12.11% and 11.57% respectively, the lowest expected yield for this bond (assuming that it does not default) is the YTM rate of 8.23%. Consequently, the yield to worst is equal to 8.23%.

Calculate the discount margin measure for a floater and explain the limitation of this measure.

Since the coupon rate on a floater is constantly changing (based on reference rate changes—LIBOR, etc.) it is difficult to determine future cash flows and calculate a precise YTM. However, there are various ways to assess the margin or spread for floaters. The most popular technique is the **discount margin**, which estimates the average margin over the reference rate that the investor expects to earn during the life of the security.

The investor follows four basic steps:

1. Determine the cash flows assuming that the reference rate does not change over the life of the security.
2. Select a margin.
3. Discount the cash flows in Step 1 by the current value of the reference rate plus the margin selected in Step 2.
4. Compare the present value of the cash flows in Step 3 to the price plus accrued interest. If the present value is not equal to the price, reevaluate the cash flows using a different margin. Stop the process when the margin equals the present value.

Example: Given the table below of discounted cash flows, determine the spread for a bond with a price of 99.522.

Answer: Given the table, there is only one series that corresponds with a present value of cash flows equal to 99.522 and that is the series with a 94 basis point spread. In other words, when the cash flows are discounted using a margin over LIBOR ranging from 82 BP to 98 BP, only one set of cash flows will correspond with the price in the marketplace. In this example, the discount margin happens to be 94 BP.

	Present value of the cash flows assuming a margin of (over LIBOR):				
	82 BP	**86BP**	**90 BP**	**94 BP**	**98 BP**
Present Value of accumulated cash flows	100.000	99.761	99.633	99.522	99.167

There are two basic problems with the discount margin technique:

- The measure assumes that the reference rate will not change over the reference rate.
- This approach does not take into consideration a floating-rate security cap or floor.

  Note: for a security selling at par, the discount margin equals the quoted margin in the coupon reset formula. Moreover, if the YTM for the bond is provided, the discount margin is simply the difference between the YTM and the corresponding reference rate.

  Example. Cyrus Vantey purchased a 6% coupon, 8-year bond with a YTM of 8.93%. What is the discount margin assuming the LIBOR reference rate is currently 6.37%?

  Answer. Do not be fooled by the irrelevant coupon or term information. The only relevant data is the YTM of 8.93% and the LIBOR rate of 6.37%. The discount margin is the difference between LIBOR and the YTM or 256 basis points (8.93 – 6.37).

Compute, using the method of bootstrapping, the theoretical Treasury spot rate curve, given the Treasury par yield curve.

- The Treasury **par yield curve** represents the adjusted on-the-run Treasury yield curve where the coupon issues are at par value and equal the YTM.

Bootstrapping refers to the process of creating a theoretical spot rate curve. Bootstrapping needs to be performed for those periods in which a rate is not available. Thus, the application of interpolation or bootstrapping enables an analyst to determine the implied rate for a specific time period. Bootstrapping is a time consuming process. It is periodically on the Level I exam and if so, is one of the more complicated calculations required. The following example demonstrates the bootstrapping process.

Question: Using the hypothetical Treasury Par Yield Curve below, determine the theoretical 2.0-year spot rate:

Period	Maturity	Annual YTM BEY (%)	Spot Rate BEY (%)
1	.5	0.030	.03000
2	1.00	0.033	.03300
3	1.50	0.035	.035053
4	2.00	0.039	.039164 (assume this is unknown)
5	2.50	0.044	.044376

Solution:

- Before starting the cash flow process, the analyst should confirm what is being asked. If the table was shown precisely as it appears above, then the analyst would know that the 2-year spot rate is 3.9164%. However this would be too easy. In order to require bootstrapping, we must assume that the one rate that we need is not available. In fact, all of the other rates are available, except the one rate that we need. Why does this happen? Sometimes, investors want to hold a bond for a period of time (2 years, 3 years, 4 years, etc.) and the spot rate does not exist for their desired time period. However, if the prior period spot rates are known, and if the purchase price is known, the implied spot rate can be determined. It requires the analyst to discount each cash flow based on the known spot rate for each corresponding period and then to find the missing piece.

- In this example, the analyst needs to find the 2-year spot rate. If we assume that all of the prior three spot rates are available, then we can discount each cash flow by the available rates. But what cash flows do we discount? In this case, the 2-year spot rate corresponds with a 3.9% Treasury. Since this is a hypothetical Treasury Par Yield Curve, each bond sells for par and offers the coupon corresponding with the annual YTM. This means that the 2-year spot rate, which offers a 3.9% YTM, has a coupon of 3.9% and sells for 100. Alternatively, if all of the cash flows are discounted at 3.9%, the value of the discounted cash flows should equal par or 100. This is shown in the table below. The cash flows of this bond equal 1.95 in each period. If each 1.95 cash flow is discounted at the 3.9% YTM rate then the discounted cash flows (including principal repayment in the final or fourth period) will equal the bond's price. The final value will equal 100.

Period	Coupon rate (semiannual period)			Payment
.5 year	0.039 * 100	* .5	=	$1.95
1.0 year	0.039 * 100	* .5	=	$1.95
1.5 years	0.039 * 100	* .5	=	$1.95
2.0 years	0.039 * 100	* .5 +100	=	$101.95

Discounting each of the cash flows at 3.9% provides a discounted cash flow equal to 100.

- The value of this bond should also equal the discounted cash flows of each coupon payment discounted at the prevailing spot rate. Consequently the price (in this case equal to par or 100) should equal the first cash flow discounted at the 1 period spot rate, the second cash flow discounted at the 2 period spot rate and so forth. This is a more time consuming process, but ensures that arbitrage opportunities do not exist. If the cash flows discounted at each spot rate provided a value other than the price shown in the marketplace then arbitrageurs could make a profit on the difference (see section on arbitrage-free bond in Chapter 5 or reconstitution).

- The cash flows for each period apply the spot rates associated with each respective period. Analysts need to be careful to take one-half of the spot rate (corresponding to the time period of the cash flow) since spot rates are quoted on an annualized bond equivalent rate (BEY).

- In this case, the discounted cash flows for this par bond equal .03/2, 033/2, and .035053/2 for periods 1 through 3. The price of 100 equals the 1.95 coupon payment divided by the spot rate for each period. Since the spot rates correspond with different time periods, the cash flows need to incorporate the respective discounting for the compounded number of years. Thus, the cash flow in the second period is the coupon payment of 1.95 divided by $(1.0165)^2$. The spot rate is taking to the 2nd power since it arrives in the second period. The third period cash flow (1.95) is divided by $(1.017527)^3$, with the third power representing the 3rd period of compounding. What is left of this puzzle? Since we do not know the fourth period spot rate (presumed to be missing in this case), but we do know the purchase price (par or 100), we only have one missing piece. This is shown below:

$$\frac{1.95}{(1.015)^1} + \frac{1.95}{(1.0165)^2} + \frac{1.95}{(1.017527)^3} + \frac{101.95}{(1+z_4)^4} = 100$$

$$1.92 + 1.89 \quad 1.85 + ??? = 100$$

Solving for the theoretical 2-year spot rate requires a little algebra. Since the price of the bond equals the discounted cash flows of each of the four terms, the fourth period cash flow *must* equal the purchase price (100) less all of the other terms (i.e. 1.92 + 1.89 + 1.85). Therefore, the fourth period cash flow must equal 94.34 (100 – 5.66). Since the final cash flow arrives in the fourth period the formula appears as follows:

$$\frac{101.95}{(1+z_4)^4} = 94.3407$$

$$(1+z_4)^4 = \frac{101.95}{94.3407}$$

$z_4 = 0.019582 = 1.9582\%$

Do not be intimidated by this formula. The computation is very easy. Remember the 94.3407 represents the total discounted cash flows less the discounted cash flows from periods 1, 2 and 3. The final payment is a future value arriving in the fourth period. Using an HP 12C, the computation is:

PV = 94.3407
FV = 101.95
N = 4
Solve for I = 1.9582 *2 = 3.9164%

- Doubling the rate of 1.9582% provides the bond equivalent rate of 3.9164%. In this case, the table provides the 2-year spot rate. However, the method of bootstrapping assumes that a particular rate is not provided. In such a case, the spot rate has to be determined through this bootstrapping methodology.

Compute the value of a bond using spot rates.

The value of a bond is equal to the discounted cash flows of each payment. When entering the data into a normal bond calculation, the analyst usually uses a YTM rate which implies the same rate each period. However, in reality, the spot rate varies for many of the periods according to the *term structure* of interest rates. In other words, the YTM rate represents a blended rate or weighted average of variable spot rates. Since a spot rate is the rate applied to a single cash flow to be received in the future, discounting the cash flows by this rate provides a very accurate value of all future cash flows. An analyst can apply either the YTM rate or each individual spot rate to future cash flows. When the cash flows are discounted using these rates they ought to equal the price of the bond. If there are differences between the discounted cash flows and the price of the bond, there may be arbitrage opportunities. However, in the absence of arbitrage, the value of the bond should equal the price as shown in the prior example. For example, the value of the bond below equals the 3.9% coupon (1.95 per period) discounted at each of the various period spot rates (e.g. 3%, 3.3%, 3.5053% and 3.9164%). The discounting in the first period uses a rate of 3% (1.5 on semiannual basis), the second period uses an annualized rate of 3.3% (3.3/2 = 1.65%) and so forth. The final computation of the bond price appears as:

$$\frac{1.95}{(1.015)^1} + \frac{1.95}{(1.0165)^2} + \frac{1.95}{(1.017527)^3} + \frac{101.95}{(1+.019582)^4} =$$

1.921 + 1.887 + 1.851 + 94.361 = 100

Notice, that this computation provides a value of par corresponding with the new issue market. This would also be the case if all of the cash flows were discounted at

3.9% (rather than 3%, 3.3%, 3.5053%, etc.). Remember that in the final period (i.e. fourth period) represents not only the coupon of 1.95, but also the return of principal (100). Thus, while the discounting may, at first, appear to be disproportionate (since the first three cash flows are below 3.9%), in actuality it works out to a par value since the final payment is very large relative to the other 3 period ash flows. The final discounted cash flows equal to 100, confirms the par yield curve.

Explain the limitations of the nominal spread.

The nominal spread represents the number of basis points separating the YTM of the non-Treasury bond and the comparative Treasury bond. To compute the nominal spread simply subtract the Treasury Bond yield from the YTM of the non-Treasury.

There are two basic problems with the nominal spread. These include:
1. The nominal spread does not incorporate the term structure of spot rates,
2. In a volatile interest rate environment, callable and putable bonds alter the cash flows of the non-Treasury bond.

Example:

An IBM, 8.5%, 10 year bond offers a YTM of 9.75%. The comparable YTM of a Treasury has a yield of 8.37% and the comparable bonds in IBM's industry have a yield of 9.68%. What is the nominal yield spread?

Solution: Do not be mislead by extraneous information. The comparable bond yields in the industry are a trap as are the inclusion of the coupon rate. The nominal yield spread is the difference between the YTM of the IBM bond 9.75% and the comparable treasury of 8.37%. The nominal spread is therefore 138 basis points (9.75% - 8.37%).

Describe the zero-volatility spread and explain why it is superior to the nominal spread.

The **Z-spread** or **zero-volatility spread** (also known as the **static spread**) measures the spread that the investor realizes over the entire Treasury spot rate curve if the bond is held to maturity. Remember that the YTM represents the blended rate that the investor receives on the entire series of cash flows. Since in any given period the spot rate differs from the YTM, the spread will also vary. The Z spread is the constant spread that is added to each risk-free spot rate (i.e. the Treasury spot rate). This is a trial-and-error method that determines the effective spread of the non-Treasury based on a differential over the Treasury spot rate curve. A series of discount rates are used until the final spread equals the non-Treasury bond price.

The difference between a Z spread and the nominal spread do not differ significantly for a bullet bond (single payment of principal) with short maturity and relatively flat yield curve. However, there will be larger differences when the yield curve is steep and when the bond is an amortizing security. In this latter case, the Z curve provides a more accurate assessment of the true yield differential.

Explain how to compute the zero-volatility spread, given a spot rate curve.

The zero-volatility spread is computed on a trial and error method. A spread is added to the spot curve and the cash flows are discounted back until they equal the value of the bond. The spread is increased or decreased until the discounted cash flows correspond to the price. If the discounted cash flows are too high, then the spread is increased. If the discounted cash flows are too low, then the spread is decreased.

Example: What is the Z spread for an 8%, 10-year bond that has a price of 104.21, priced to yield 7.4%? Assume that a comparable Treasury has a yield of 6%.

	Coupon		8%			
		Cash	Spot	Present Value ($) assuming a spread of		
Period	Years	Flow	Rate	100	125	146
1	0.5	4	0.03	3.9216	3.9168	3.9127
2	1	4	0.033	3.8334	3.8240	3.8162
3	1.5	4	0.035053	3.7414	3.7277	3.7163
4	2	4	0.039164	3.6297	3.6121	3.5973
5	2.5	4	0.044376	3.4979	3.4767	3.4590
6	3	4	0.04752	3.3742	3.3497	3.3293
7	3.5	4	0.049622	3.2565	3.2290	3.2061
8	4	4	0.05065	3.1497	3.1193	3.0940
9	4.5	4	0.051701	3.0430	3.0100	2.9825
10	5	4	0.052772	2.9366	2.9013	2.8719
11	5.5	4	0.053864	2.8307	2.7933	2.7622
12	6	4	0.054976	2.7255	2.6862	2.6536
13	6.5	4	0.056108	2.6210	2.5801	2.5463
14	7	4	0.056643	2.5279	2.4855	2.4504
15	7.5	4	0.057193	2.4367	2.3929	2.3568
16	8	4	0.057755	2.3472	2.3023	2.2652
17	8.5	4	0.058331	2.2596	2.2137	2.1758
18	9	4	0.059584	2.1612	2.1148	2.0766
19	9.5	4	0.060863	2.0642	2.0174	1.9790
20	10	104	0.062169	51.1835	49.9640	48.9632
				107.5416	105.7166	104.2146

Solution: In this example, the table shows the same cash flows in column 3 (i.e. 4 or the semiannual coupon) corresponding to different discount rates. The discount rates are determined on a trial and error method using the prevailing spot rates at the time. For example, in the 5^{th} period the spot rate is 4.4736%. The trial and error method adds a variety of spreads to this spot rate (i.e. 100, 125 and 146 BP) to this spot rate and discounts the cash flows for each period (the numbers below 100, 125 and 146). When all of the discounted cash flows are added together they arrive at a price shown at the bottom of the cash flows (i.e. 107.5416, 105.7166 or 104.2146). The appropriate Z spread corresponds with the price shown in the marketplace. Thus, if an analyst sees a price of 104.21, the Z spread must be 146 basis points. This is the price determined when using the set of discounted cash flows. Note: in this case the nominal spread was 140 BP. The nominal spread was simply the difference between the YTM of the Treasury and the YTM of this bond (7.4% - 6%).

Explain why the zero-volatility spread will diverge from the nominal spread.

The zero volatility spread diverges from the nominal spread whenever the yield curve is steep or whenever the principal is repaid over time (amortizing bond). The Z-spread for a zero-coupon bond will not be affected by the term structure but a mortgage-backed security or asset-backed security (amortizing securities) in steep yield curve environment would be affected.

Explain the option-adjusted spread for a bond with an embedded option and what is meant by the option cost.

An **option-adjusted spread** (OAS) measures the yield spread that converts dollar differences between value and market price. The OAS depends on the valuation model being used. The spot rate curve that is applied is not a single curve; rather it is a series of spot rate curves that provide for changes in interest rates. The Z-spread does not factor changes in cash flows with changing interest rates. The Z-spread assumes zero interest rate volatility.

The **option cost = Z-spread – OAS**

The OAS cost is positive for callable bonds and most mortgage-backed and asset-backed securities (Z-spread is greater than OAS) and negative with putable bonds (OAS is greater than Z-spread). In the former, the OAS is less than the Z-spread owing to the borrower's ability to alter cash flows and in the latter case the OAS is greater than the Z-spread (OAS cost is negative) because of the investor's ability to alter cash flows. In other words, the investor demands a greater return for a callable bond (because the call risk works against the investor) and is willing to pay more money (i.e. demand lower return) for a put option (because a put option reduces risk and is considered desirable by the investor). Consequently, the option value has a value greater than 0 with callable bonds (positive value that the seller has to provide to the buyer) and a value less than 0 (the investor has to pay for this value).

Example: Suppose that the Cisco company has a callable bond with an OAS of 187bp. Similar option-free bonds have a Z-spread of 149bp. What is the cost of this callable option?

Solution: The OAS – Z spread of an option free bond equals 187 – 149 or a cost to the option of 38 basis points.

Illustrate why the nominal spread hides the option risk for bonds with embedded options.

The Z-spread = OAS + option cost, whereas

The nominal spread approximates OAS + option cost

A high nominal spread could be concealing a high option cost. For example, a nominal spread could be 250 basis points, with an option cost of 220 and an OAS only 30 BP. An investor is only compensated for the OAS and may not be adequately compensated for taking a security with an embedded option if relying only on the nominal spread.

Explain a forward rate.

Forward rates assess the market's consensus of future interest rates. The forward rates are extrapolated from the default-free theoretical spot rate curve.

In the following example, an investor should be indifferent between buying a one-year Treasury Bill or one six month Tbill and another six month Tbill upon the first bill maturity. In both cases, the investor receives an effective 9% annual return.

Implied Forward Rate Problem:

- Given the following spot rates, calculate the implied forward rate between the two periods:

Six month Tbill spot rate = .09
One-year Tbill spot rate = .095

Solution for forward rate problem:

$$f = \frac{(1.0475)^2}{1.045} - 1 = \mathbf{1.050} = \mathbf{5\%}$$

Proof:

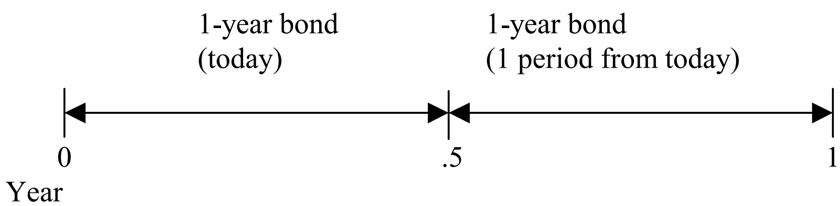
Price of One Year Tbill = $100/(1.0475)^2$ = 91.1361

Price of One Year Tbill after six months earning 4.5%

$91.1361 * 1.045 = 95.2372

$95.2372 * 1.05 (interest rate for last six months) = 100

1 Year Bond (2 periods)



In this case investors have the option to invest in 2 six month Treasuries or to invest in one 1 year Treasury. Since the six month Treasury is 9% and the 1 year spot rate is 9.5%, investors need to be indifferent between investing in six months today + another six month Treasury in six months compared to an investment today of 1 year. The forward rate question asks what is the six-month rate, six months from today? The notation is ${}_nf_m$ where n =number of periods and m = number of periods from today. Consequently, the notation ${}_1f_1$ asks what is the one period forward rate, one period from today. The notation ${}_1f_8$ asks what the 1 period forward rate is eight periods from now.

Explain and illustrate the relationship between short-term forward rates and spot rates.

The forward rates are computed by taking the spot rate for the next period to the nth level (power = number of periods) divided by the current spot rate to the power corresponding to its number of periods. Thus the one period forward rate, 3 periods from now (${}_1f_3$)

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represents the 3rd and 4th periods. The computation for $_1f_3$ equals $(1.039164)^4/(1.035053)^3$. This computation is 1.1661/1.10889 = 5.16% as shown below as the forward rate in the third period looking forward.

Compute spot rates from forward rates.

A spot rate is comprised of a series of forward rates. For example, the value of a 3-year spot rate will depend on the current 6-month spot rate and the five 6-month forward rates. The 3-year spot rate actually equals the geometric average of the current 6-month spot rate and the five 6-month forward rates. For example, given the table below the spot rate of 4.752% = [1+.030/2 (first period rate) * 1+036/2 * 1 +.0392/2 * 1+0516/2 * 1+06549/2 * 1+06338/2]. The forward rates are divided in half since for purposes of quotation they are described in the annualized bond equivalent yield (BEY) manner. Consequently the actual calculation confirming the 3-year spot rate of 4.752% = $[(1.015 * 1.018 * 1.01958 * 1.02577 * 1.03268 * 1.03166)]^{1/6} - 1 = .023761$ = 2.3761% * 2 = 4.7522%. Note: if the forward rates were multiplied $(1.03 * 1.036 * 1.0392 * 1.0516 * 1.06549 * 1.06338)^{1/6}$ = 4.752%

Period	Maturity	Annual Spot YTM BEY (%)	Rate BEY (%)	Forward Rate (BEY) (%)
1	.5	3.0	3.0	1.067/1.03=3.6%
2	1.00	3.3	3.30	1.10889/1.0671=3.92%
3	1.50	3.5	3.5053	1.1661/1.10889=5.16%
4	2.00	3.9	3.9164	1.24247/1.1661=6.549%
5	2.50	4.4	4.4376	1.32122/1.24247=6.338%
6	3.00	4.7	4.752	

Problems: Fabozzi, Chapter 6

1. What is the approximate YTM of a bond with a current market price of $850, annual coupon of 10%, with 5 years to maturity?
   A. over 14%
   B. between 13% and14%
   C. between 12% and 13%
   D. less than 12%

2. What is the approximate YTM of a bond with a 12% coupon, 10 years to maturity and selling at 88?

   A. over 14%
   B. between 13% and14%
   C. between 12% and 13%
   D. less than 12%

3. What is the AYC of a 14%, 15 year bond, trading at 115 with 5 years remaining to first call and a call price of 112?
   A. over 13
   B. between 12% and13%
   C. between 11% and 12%
   D. less than 11%

4. Assume that you purchased a 9%, 10-year bond for $800. In the event that interest rates decline from 12.6% (the implicit market interest rate) to 11% during the next two years, your bond will increase in price to $897. What is the approximate realized yield that you would receive during the next two years?
   A. over 17
   B. between 16% and17%
   C. between 15% and 16%
   D. less than 15%

5. Assume that you purchased a 9%, 10-year bond for $800. Confirm that the implied market interest rate on this bond is 12.6%? Next, confirm that the bond price rises to $897 when market interest rates drop to 11% in two years. What is the bond price if interest rates decline in two years to 10% (rather than 11%), and what is the horizon yield for 2 years?
   A. over 19%
   B. between 17% and 18%
   C. between 16% and17%

D. between 15% and 16%
E. less than 15%

6. For a callable bond, yield-to-call is a more conservative measure of yield whenever:

A. the bond is priced at or above its call price
B. the bond is trading for more than par but less than the call price
C. the bond is trading at less than par
D. none of the above

7. When computing yield to maturity, the implicit reinvestment assumption is that interest payments will be reinvested at the:

A. prime rate prevailing throughout the life of the loan
B. coupon rate prevailing throughout the life of the loan
C. market interest rates prevailing when payments are made
D. computed yield-to-maturity

Please use the following information for the next two questions:

Par value	$1000
Time to maturity	20 years
Coupon	10% (with interest paid annually)
Current price	$850
Yield to maturity	12%

8. Intuitively and without the use of calculations, if interest payments are reinvested at 10%, the realized compound yield on this bond must be:

A. 10%
B. 10.9%
C. 12.0%
D. 12.4%

9. Given the bond described above, if it paid interest semiannually (rather than annually), but continued to be priced at $850, the resulting yield-to-maturity would be:

A. less than 10%
B. more than12%
C. less than 12.0%
D. 12%

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E. cannot be determined

10. Given the following theoretical spot rates for a 10-year treasury, what is the spot rate for a 2.5-year maturity?

Maturity		Coupon Rate	YTM	Spot	Price Rate
.5	0.000	0.080	.08		$96.15
1.00	0.000	0.083	.083		$92.19
1.50	0.085	0.089	.0893		$99.45
2.00	0.090	0.092	.09247		$99.64
2.50	0.11	0.094	????		$103.49

A. 8.930%
B. 9.247%
C. 9.321%
D. 9.468%
E. 9.874%

11. What is the bond-equivalent yield of an annual-pay bond that has a yield to maturity of 7%?
A. 6.88%
B. 7.23%
C. 6.47%
D. 3.5%

12. What is the nominal spread for the bond in the table below?

Issue	Coupon	Price	YTM
Treasury	6%	100	6%
Non-Treasury	8%	104.19	7.4%

A. 120 BP
B. 200 BP
C. 419 BP
D. 140BP

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13. What is the Z-spread for an 8%, 10 year bond that has a price of 105.71? Assume that the comparable Treasury has a YTM of 6% and the bond below is priced to yield 7.4%.

    A. 140 BP
    B. 146 BP
    C. 571 BP
    D. 125 BP

Answers: Fabozzi, Chapter 6

1. **A**. Approximate YTM answer: (100+150/5)/(1850/2) =

   130/925 = 14.05%

   [Actual YTM =14.41%]
   The approximation approach serves as a handy reference rate when the investor does not have access to a financial calculator. Moreover, the approximate YTM also provides a convenient reference point to make sure that the investor correctly calculates the actual YTM.

   Note: YTM for a zero coupon bond is a special case in which the issuer only makes a single payment The approximate YTM calculation for a zero coupon bond is not as accurate as a coupon bond, but the calculation is shown as follows:
   Approximate YTM Example: current price = $400, zero coupon bond,
   (Zero coupon) maturity = 10 years
   Approx. YTM Answer (0 + 600/10)/(1400/2) = 8.57%
   [Actual YTM = 9.6%]

2. **A.** Approximate YTM (120+12)/940 = 14.05%
   actual YTM = 14.33%

3. **C**. AYC = $\dfrac{140 + (1120-1150)/5}{(1120 + 1150)/2}$

   $\dfrac{140-6}{1135}$ = 11.81%

   [Actual YTC = 11.76%]

4. **B**. $\dfrac{90 + (897-800)/2}{(897+800)/2}$

   138.5/848.5 = 16.3%

   [Actual return = 16.8%]

5. **A**. coupon= $90
   term = 8 years

maturity price = $1000

The price in 2 years (with 10%)= $946.7, horizon yield =19.6% annualized yield

6. **A**. The call price becomes a more conservative value whenever the bond price is above the call price.

7. **D**. The YTM is the reinvestment rate assumption.

8. **B.** Since the YTM is 12% and the reinvestment rate is 10% the total return must be in between these two rates. Answer "B" is the only intuitive answer that falls between 10% & 12%.

9. **C.** The YTM decreases with increased frequency of payments.

10. **D.** The solution for the bootstrapping problem follows:

$$\frac{5.50}{(1.04)^1} \quad \frac{5.50}{(1.0415)^2} \quad \frac{5.50}{(1.04465)^3} \quad \frac{5.50}{(1.046235)^4} \quad \frac{105.50}{(1+Z_5)^5} = 103.49$$

5.288 + 5.07 + 4.824 + 4.59 + x = 103.49

83.718 = PV
105.50 = FV
n = 5
I = ?

I = 4.734 * 2 = **9.468**

11. **A.** The bond-equivalent yield is always less than the annual-pay YTM. The computation follows: $2[(1 + .07)^{0.5} - 1] = 6.88\%$

12. **D.** The nominal spread for this bond is 140 basis points. It is the difference between the YTM of the Treasury and the non-treasury bonds.

13. **D**. The Z-spread corresponding to the price of 105.71 is 125 basis points. The Z-spread is the trial and error method, which adds the same spread to each spot rate throughout the entire series of cash flows. Thus, the price of 105.71 is determined by discounting each cash flow (principal and interest) for the 20 periods by the spot rate plus 125 basis points.

Introduction to the Measurement of Interest Rate Risk

Chapter 7, Fabozzi

Introduction

Interest rate risk deals with the impact of interest rate changes on the price level of a bond. Since interest rates are inversely related to bond prices, increase in interest rates cause bond prices to fall while decreases in interest rates cause bond prices to rise. Bonds with embedded options may not cause the bonds to behave differently. Moreover, changes in interest rates do not cause all bonds to behave in the same manner. Some bonds, perhaps owing to differing coupon payments, term to maturity, credit risk, or some other factor may change faster or slower compared to other bonds. It is this price volatility or interest rate risk that Chapter 7 explores. This material lends itself to numerous short problems, well suited to Level I exam questions.

Distinguish between the full valuation approach and the duration/convexity approach for measuring interest rate risk, and explain the advantage of using the full valuation approach.

The primary focus of interest rate risk is measuring the impact after an adverse rate change. There are two approaches to measuring interest rate risk: the full valuation approach and the duration/convexity approach.

The **full valuation approach** also known as **scenario analysis** examines the value of bonds under a variety of interest rate scenario changes. For example, a portfolio manager might examine the change in a bond with assumed interest rate increases of 50, 100,150 and 200 basis point increases and decreases. This approach is useful when there is a good valuation model and can be used for parallel and nonparallel shifts in the yield curve.

Highly leveraged investors (such as hedge fund investors) often use extreme scenario tests, known as **stress testing**, to examine the impact of wide interest rate changes. This is fine so long as the manager has a good valuation model to estimate what the price of the bonds will be in each interest rate scenario.

The advantage of the duration/convexity measure is that it is a simpler measure that will show how a portfolio or single bond will change if rates change in a parallel fashion.

Compute the interest rate risk exposure of a bond position or of a bond portfolio, given a change in interest rates.

Assume a bond with a 6% coupon has a five-year maturity in any environment with 6% market interest rates. In this case the bond will have a par price. Now suppose that market interest rates drop 100 BP to 5% (from 6%). The price will rise to 104.376. The price will drop to 95.8417 if market interest rates rise 100 BP to 7% (from 6%). Thus, a 100 BP rise or fall will cause the bond price to move from par to either 104.376 or 95.8417. These changes occur in an environment with parallel shifts in the yield curve. Of course, if there are nonparallel shifts in the yield curve these results may not hold.

Explain why it is difficult to apply the full valuation approach to a bond portfolio with a large number of positions, especially if the portfolio includes bonds with embedded options.

In an environment with a large number of bonds, if even a few are complex (i.e. with embedded options) the full valuation process with multiple scenarios becomes extremely time-consuming. Not all bonds will move in the same direction with the same level of movement. Embedded options may cause some bonds to rise or fall, independent of interest rate changes. Moreover, some bonds will have greater exposure to interest rate movements due to their credit risk, term to maturity and amount of coupon payment. Therefore, the full valuation approach with a portfolio of bonds, as implied in the question, is a difficult process. The duration/convexity approach is a starting point for gauging interest rate volatility.

Explain and illustrate the price volatility characteristics for option-free bonds when interest rates change (including the concept of "positive convexity").

As the required yield increases, the price of an option-free bond decreases. However, the relationship is not linear (straight line relationship). The shape of the price/yield relationship is convex as explained below:

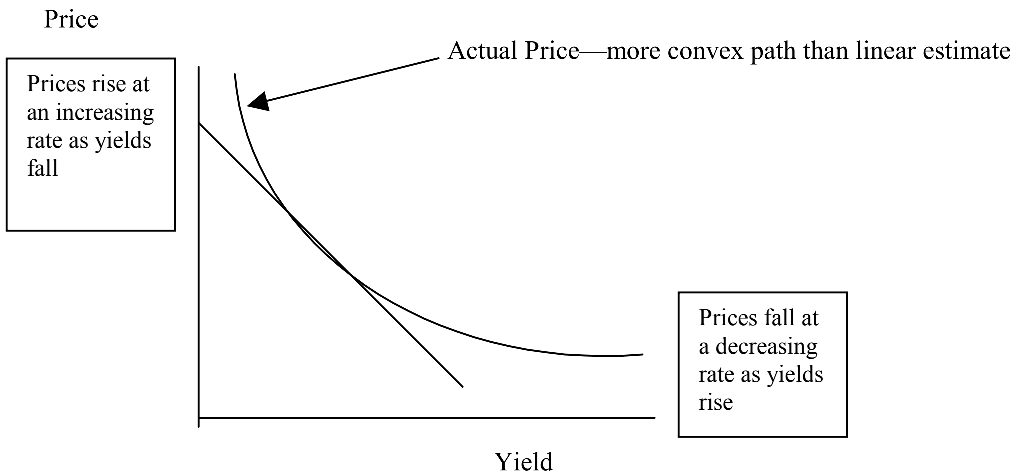
The market price of a bond is a function of four factors: 1) par value; 2) coupon; 3) number of years to maturity; 4) prevailing market interest rate. Moreover, the following relationships shown below, between yield (interest rate) changes and bond price behavior hold true. Level I candidates need to remember these relationships, they frequently appear on an exam.

Key Relationships

- Bond prices move inversely to interest rates
- For a given change in interest rates, longer-term bonds have larger price changes
- Price volatility increases at a diminishing rate as term to maturity increases
- Price movements resulting from equal absolute increase or decreases in yield are not symmetrical. Decreases in interest rates result in increases in bond prices by more than an increase in interest rates on the same amount lowers prices.
- Higher coupon issues demonstrate smaller percentage price fluctuation for a given change in interest rates; thus suggesting that bond price volatility is inversely related to coupon.

Positive Convexity—Option Free Bond

The curved shaped benefits an investor with interest rate changes (compared to a linear relationship)



The key point to remember with positive convexity, illustrated by the drawing above, is that the actual price is always above the linear relationship estimated by duration (with an option-free bond). It is this difference that convexity attempts to correct. Moreover, the more significant the curve, the better for the investor, this will be an indication of greater convexity. Notice that as the curve becomes more bent at the top (e.g. left side of drawing), the actual price is higher than implied by the linear estimate. Similarly, if the curve becomes more bent on the lower or right side, it will be higher than the linear estimate. In both cases, having a larger degree of convexity (i.e. bend in the price curve) suggests that the investor has an improved condition relative to the linear estimate (shown by straight line). Thus, positive convexity is good for an investor, and worthy of a higher price.

❖ **Example:** Demonstrate why price movements resulting from equal absolute increase or decreases in yield are not symmetrical. For instance, show how decreases in interest rates result in increases in bond prices by more than an increase in interest rates on the same amount lowers prices.

Solution: Take a 10-year 7% bond trading at par. Now increase and decrease the market interest rates by 1%. At a 6% YTM the bond trades for 1,074.387 or a change of 7.44%. However, at market interest rates of 8% the bond trades for 932.05 or a reduction of 6.8%. Thus, an equal change in market interest rates (i.e. 1%) caused a greater price increase than decrease.

Proof:
Rate change from 7% to 6%
FV =1000, N =20, I =3%, PMT = 35, solve for PV = 1074.387
(1074.387 – 1000)/1000 = 7.44% increase

Rate change from 7% to 8%
FV =1000, N =20, I =4%, PMT = 35, solve for PV = 932.05
(932.05 – 1000)/1000 = 6.8% decrease

Explain and illustrate the price volatility characteristics of callable bonds and pre-payable securities when interest rates change (including the concept of "negative convexity").

CONVEXITY

The curved nature of the price-yield relationship is known as the convexity of the bond. Technically, it is also known as the second derivative of the price-yield relationship or the change of the change. Since modified duration is an estimate of the price-yield relationship, there is often an error when there are large yield changes. The error increases (the curvature becomes more distinct) with increasing rate changes. The differences in the curvature relative to the linear estimate are estimated by the convexity. The following convexity relationships should be clear:

1. there is an inverse relationship between coupon and convexity

2. there is a direct relationship between maturity and convexity

there is an inverse relationship between yield and convexity (causing a more convex relationship at the lower-yield end of the curve).

CALLABLE BONDS

- Callable bonds have a negative convexity feature built in.

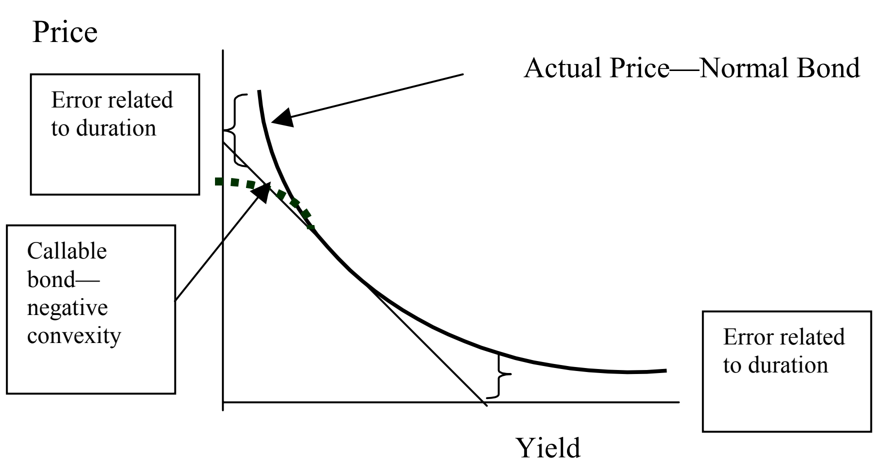
- This is an unattractive feature that limits the upside potential of the bond, and causes a compression in bond price.

Although there are various ways to view the call feature of the bond (such as the call bond price = noncall price - call option price), the Level I candidate should primarily focus on the "non-convex" shape of the yield curve.

The two most common types of embedded options are call (or prepay) options and put options. As interest rates decline, the issuer may prepay or call the debt obligation prior to the scheduled principal repayment date. There is little incentive to call a bond when market interest rates are above the coupon rate. The likelihood of calling or prepaying a bond increases as market interest rates drop below the coupon rate. This limits the price appreciation relative to the price decline given a comparable change in market interest rates leading to **negative convexity**. Option-free bonds, on the other hand, experience **positive convexity** in which price increases (associated with market interest rate declines) exceed price decreases (associated with market interest rate increases) for a comparable change in market interest rates.

A putable bond enables the investor to put the bond back to the issuer at a predetermined price. Presumably the value of the putable bond is greater when interest rise since the bondholder can put the bond back to the issuer when the price would otherwise decline. Thus, the value of the putable bond is equal to the option-free bond plus the value of the put.

Callable bond—negative convexity



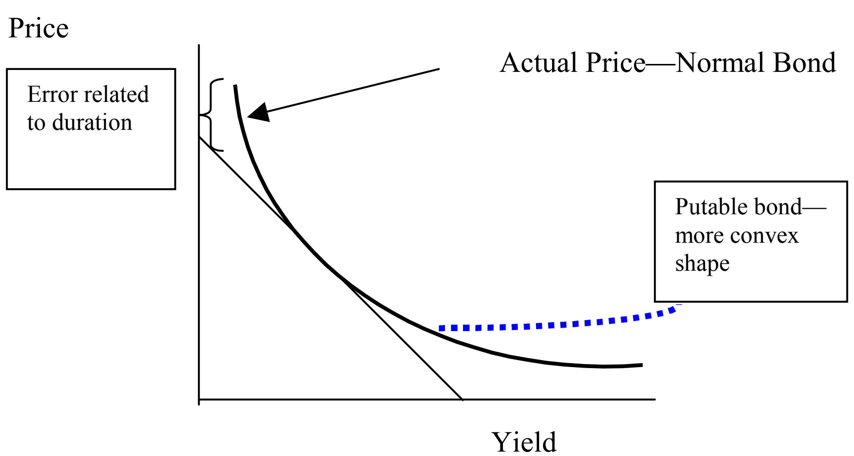
Convexity is the percentage price change not explained by duration.

Large changes in yield should include both the estimates given by duration and convexity.

Convexity Properties:

- As the required yield increases (decreases), the convexity of a bond decreases (increases).
- For a given yield and maturity, the lower the coupon, the greater the convexity of a bond.
- For a given yield and modified duration, the lower the coupon, the smaller the convexity.
- Positive convexity refers to the absolute and percentage price changes being larger for yield declines than with yield increases of the same number of basis points. The normal yield curve represents positive convexity.
- Negative convexity refers to the absolute and percentage price changes being larger for yield increases than with yield decreases of the same number of basis points. The callable bond demonstrates negative convexity—with yield declines the bond price levels off (since the bond may be called with lower market interest rates).
- The convex shape of the yield curve suggests that the actual outcome is more desirable than the linear estimation of duration. A general rule of thumb, the more convex the curve the better (often referred to as the "benter the better"). This is why a putable bond is more preferable to a normal option-free bond or a callable bond.

Putable bond—more convexity



A putable bond does not decline as much as a normal option-free bond with increases in interest rates. Thus, it is more desirable for the investor. This increased convexity is attractive to investors since it provides more downside protection during periods of rising interest rates. Investors find having downside protection a very attractive feature and are willing to pay more for this increased convexity.

Compute the duration of a bond, given information about how the bond's price will increase and decrease for a given change in interest rates.

Duration is a measure of the approximate sensitivity of a bond's value to rate changes. Specifically, it is the approximate percentage change in value for a 100 basis point change in rates.
Duration without the formulas (assuming the only available information is the pricing of the bond with modest increases and decreases in market interest rates)

Duration and convexity can be calculated easily without formulas. The steps are shown below:

1. Increase the yield on the bond by a small number of basis points and calculate the new price at the higher interest rate. This price will be referred to as V_+

2. Decrease the yield on the bond by the same number of basis points and calculate the new price. This price will be referred to as V_-

3. Assume that V_0 is the initial price, duration can be approximated using the following formula

$$\text{Approximate duration} = \frac{V_- - V_+}{2\, V_o(\Delta y)}$$

Assume the following:
Δy = 20 BP (starts at 6%)
V_o = 134.6722 (9% coupon, 20-year option free bond)
V_- = 137.5888
V_+ = 131.8439

$$\text{Duration} = \frac{137.5888 - 131.8439}{2 * (134.6722) * (.002)} = 10.66$$

Question: Given the following information, compute the approximate duration.

Current Bond Price (yield at 7%)	Bond price with yield at 7.2%	Bond Price with yield at 6.8%
113.47	110.86	116.45

Solution: Approximate duration = $\dfrac{V_{-} - V_{+}}{2\,V_0(\Delta y)}$

$$\frac{116.45 - 110.86}{2 * 113.47 * (.002)} = 12.3155$$

Compute the approximate percentage price change for a bond, given the bond's duration and a specified change in yield.

The formula for a modified duration enables an analyst to estimate a price change, given an expected change in market interest rates. The formula is:

Approximate % price change of a bond = (-)(duration)*(Δy)

Where duration represents modified duration and
Δy represents the change in market interest rates

Consequently, the (modified) duration of 10.66 implies that for a 100 BP change in market interest rates the approximate change in this bond price will be 10.66%. The duration estimate

Characteristics of duration:

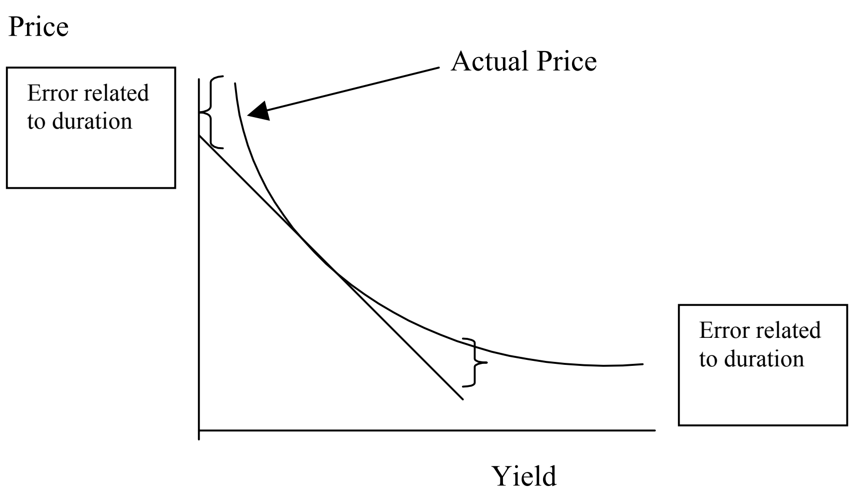
- The duration of a bond with coupon payments will always be less than its term to maturity (since duration gives weight to interim coupon payments).
- There is an inverse relationship between coupon and duration. A bond with a larger coupon will have a shorter duration because more of the total cash flows come earlier in the form of interest payments.
- A zero coupon bond will have a duration equal to its maturity. This is the case because there is only a single payment.
- A positive relationship exists between the term to maturity and duration. Duration increases at a decreasing rate with maturity.
- The shape of the duration-maturity curve depends on the coupon and the yield to maturity.
- There is an inverse relationship between YTM and duration. A higher YTM reduces duration

- Sinking funds and call provisions can have a dramatic effect on a bond's duration. These provisions can significantly accelerate the total cash flows and therefore significantly reduce the bond's duration.

Example: Jared Symington had a 7%, 6-year bond with a YTM of 9.37%. Jared's bond has a duration of 11.74 and a convexity of 27.99. He is concerned about market interest rising in the near future. How much will his bond price change if market interest rates increase by 100 BP?

Solution: Do not be fooled with the extraneous information. The coupon, term, YTM and convexity are all traps! This is a simple question requiring a simple computation. Take the duration of 11.74 and multiply by –1%. His bond will drop by 11.74%. Note that duration is a linear estimate. If interest rates had been expected to decline by 1%, the bond would be expected to rise by 11.74%. In actuality, the same interest rate change does not affect price increases and decreases equally. The notion of convexity causes interest rate declines to cause bonds to rise by more than the same interest rate increase will cause bonds to fall.

Explain, using both words and a graph of the relationship between price and yield for an option-free bond, why duration does an effective job of estimating price changes for small changes in interest rates but is not as effective for a large change in rates.



In calculating duration it is necessary to evaluate the shock of interest rates by a similar number of basis points. The impact on duration will vary depending on the size of the shock (change in interest rates) as well as the convexity of the bond. The longer the maturity and greater the convexity, the greater will be the change given a rate shock. Moreover, bonds with embedded options may be more susceptible to even small rate shocks.

Bond Price Volatility

- The graph of the price/yield relationship for an option-free bond indicates that for a given change in yield, the percentage price increase with a yield decline is larger than the percentage price decrease with a comparable yield increase.

- This percentage change works to the investor's advantage and is referred to convexity (due to the convex shape of the curve).

Price changes are more significant with lower coupons and longer-term bonds. For example:

Yield	6%--5yr	6%--20yr	9%--5yr	9%--20yr
4%	108.98	127.36	122.46	168.39
6%	100	100	112.80	134.67
8%	91.89	80.21	104.06	109.90

Key points of the above table include:

- For a given change in interest rates, low coupon bonds have a larger percentage move than high coupon bonds
- For a given change in interest rates, long maturity bonds have a larger percentage price change than short maturity bonds
- Price changes increase at a decreasing rate as term to maturity increases
- Price volatility of a bond is greater when the YTM for a bond is lower
- Prices rise at an increasing rate as yields fall and prices decline at a decreasing rate as yields rise (implication of convexity)

Example:

From the table above, which bond is likely to experience the lowest percentage rate change given a 200BP increase in market interest rates?

Solution: The bond with the highest coupon and the shortest duration will have the smallest percentage change. In the above example, this corresponds with the 9%, 5–year bond. Moreover, the smallest percentage price change occurs when interest rates move from 6% to 8%. When interest rates rise from 4% to 6%, the bond falls in price from 122.46 to 112.80. This is a change of 7.89% [(122.46 – 112.80)/122.46]. However, when market rates rise from 6% to 8%, the bond falls from 112.80 to 104.06 or by 7.75%.

Distinguish between modified duration and effective (or option-adjusted) duration.price and yield for a putable bond.

Modified duration is the approximate percentage change in a bond's price for a 100 BP change in yield (assuming the cash flows don't change when the yield changes).

When valuing bonds with embedded options, it is possible that the expected cash flows may also change. **Effective duration** or **option-adjusted duration** takes into consideration changes in expected cash flows and different interest rates. For example, if a bond has a call option or put option the cash flows will vary depending on market interest rates. The modified duration will not change as a result of market interest rate changes whereas the effective duration may change.

Example: Gerald Ismeele has a 9%, 30 year bond that he bought at par. Given the table below, determine the modified and effective duration for Gerald's bond.

Price	Current	Call Price	-100bp	+100bp	
	1000	1050	1113.17	905.35	
Modified Duration					
Effective Duration					

Modified duration = (1113.17 – 905.35)/(2*1000*.01) = 10.39
Effective duration = (1050 – 905.35)/(2*1000*.01) = 7.23

Note that the effective duration computation inserted the call price of $1050 rather than the implied price of $1113.17. The call price limits the price appreciation of Gerald's bond and causes the effective duration to be lower than the modified duration (e.g. 7.23 versus 10.39). While the question did not request information regarding convexity, this is an example of negative convexity.

Explain why effective duration, rather than modified duration, should be used for bonds with embedded options.

There may be a significant difference between modified duration and effective duration when examining bonds with embedded options. For example, a callable bond could have a modified duration of 6 but an effective duration of only 4. Moreover, with CMOs, the modified duration could be 6 and the effective duration could be 18. Using modified duration for bonds with embedded options could be misleading. Effective duration is more appropriate for bonds with embedded options. This is true since the cash flows related to effective duration may not be appropriately represented with modified duration.

Explain the relationship between modified duration and Macaulay duration and the limitations of using either duration measure for measuring the interest rate risk for bonds with embedded options.

Macaulay Duration:
Macaulay developed the concept of duration approximately 50 years ago. Macaulay demonstrated that the bond's duration was a more appropriate measure of time characteristics than the term to maturity of the bond because duration incorporates both the repayment of capital at maturity, the size of the coupon and timing of the payments. Duration is defined as the weighted average time to full recovery of principal and interest payments. The denominator is the price of a bond as determined by the present value model, whereas the numerator is the present value of all cash flows weighted according to the time to cash receipt.

$$D = \frac{\sum_{t=1}^{n} \frac{C_t(t)}{(1+i)^t}}{\sum_{t=1}^{n} \frac{C_t}{(1+i)^t}}$$

Modified duration
An adjusted measure of duration that is used to measure the price volatility of a bond is called modified duration. Modified duration equals Macaulay duration divided by 1 + current yield to maturity (divided by number of payments in one year).

Example: What is the modified duration of a bond with a Macaulay duration of 10 years, a yield to maturity of 8 percent and semiannual payments?

Solution:

$$D_{mod} = 10/(1 + .08/2)$$

$$10/1.04 = 9.62 \text{ years}$$

Modified duration shows how bond prices move proportionally with small changes in yields. Specifically, modified duration estimates the percentage change in bond price with a change in yield.

$$\frac{\Delta P}{P} \text{ x } 100 = -D_{mod} * \Delta y$$

ΔP = change in price for the bond
$-D_{mod}$ = the modified duration for the bond
Δy = yield change in basis points divided by 100
P = beginning price for the bond

Assume in the above example that the bond's YTM is expected to decline 75 basis points. Then the expected change in bond price should be:

$$\%\Delta P = -9.62 \text{ x } \frac{-75}{100} = (-9.62) \text{ x } (-.75) = 7.215\%$$

- It should be noted that both modified and Macaulay duration are inaccurate measures of a bond's price sensitivity to interest rate changes for a bond with embedded options. Both of these measures do not recognize the change in cash flows for bonds with embedded options.

Describe the various ways that duration has been interpreted and why duration is best interpreted as a measure of a bond or portfolio's sensitivity to changes in interest rates.

- Duration is known as the approximate percentage price change in price for a 100 basis point change in rates. This measure is often known as the "first derivative" of the price/yield function. However, this interpretation does not convey much information to a portfolio manager that needs to understand the affect of interest rates changes on his/her portfolio.
- Duration is often thought as a measure of time. While, the concept of duration as originally designed by Macaulay, was in temporal terms, the concept is best confined to the relationship between market interest rate changes and change in bond prices.
- Investors should simply use the term duration in the context of change in a portfolio's value given a change in market interest rates by 100 BP. Other definitions using derivatives or time will be confusing to the investor.

A portfolio's duration can be ascertained by calculating the weighted average of the duration of the bonds in the portfolio. The weight represents the relative dollar value of a security to the portfolio. A critical assumption is that the yield of each bond must change by 100 BP for the duration measure to be useful. Alternatively, the dollar price change can be calculated for each security in a portfolio and then the collective price changes can be summed.

Compute the convexity of a bond, given information about how the price will increase and decrease for a given change in interest rates;

Convexity can be calculated easily without formulas. The steps are shown below:

1. Increase the yield on the bond by a small number of basis points and calculate the new price at the higher interest rate. This price will be referred to as V_+

2. Decrease the yield on the bond by the same number of basis points and calculate the new price. This price will be referred to as V_-

3. Assume that V_0 is the initial price, convexity can be approximated using the following formula

$$\text{Approximate convexity} = \frac{V_+ + V_- - 2\,V_0}{2\,V_0(\Delta y)^2}$$

Example: Compute the approximate convexity assuming the following information and a 20 basis point change in market interest rates:
V_0 = 134.6722
V_+ = 131.8439
V_- = 137.5888

Solution:

$$\text{Convexity measure} = \frac{131.8439 + 137.5888 - 2(134.6722)}{2(134.6722)(0.002)^2} = 81.96$$

Convexity is the percentage price change not explained by duration.

Large changes in yield should include both the estimates given by duration and convexity.

Convexity adjustment to percentage price =
Convexity measure x (yield change)^2 x 100

Convexity Properties:

- As the required yield increases (decreases), the convexity of a bond decreases (increases).
- For a given yield and maturity, the lower the coupon, the greater the convexity of a bond.
- For a given yield and modified duration, the lower the coupon, the smaller the convexity.
- Positive convexity refers to the absolute and percentage price changes being larger for yield declines than with yield increases of the same number of basis points.
- The convex shape of the yield curve suggests that the actual outcome is more desirable than the linear estimation of duration. A general rule of thumb, the more convex the curve the better (often referred to as the "benter the better").

Computing the convexity adjustment to the duration estimate:

Given the convexity of the bond discussed above (81.96), the estimated adjustment related to a 200 basis point market interest rate change would be:

$81.96 * (.02)^2 * 100 = 3.28\%$

Compute the estimate of a bond's percentage price change, given the bond's duration and convexity and a specified change in interest rates.

The total price change related to a 200 BP market rate increase would be:

Estimated change using duration	=	– 21.32%
Convexity adjustment	=	+ 3.28%
Total estimated percentage price change	=	- 18.04%

The actual price change was –18.40% demonstrating the combined estimation of duration and convexity. Duration by itself projected a price reduction of 21.32%. This overestimation was adjusted by convexity to the extent of 3.28% but this adjustment still resulted in a slight differential.

Note that if interest rates declined 200BP, the combined duration and convexity measures would predict a bond increase of 24.60% (21.32+3.28). However, the actual increase was 25.04% showing a differential slightly larger than with the market interest rate increase. This is consistent with the relationship of positive convexity. Positive convexity shows gains larger than losses for a comparable interest rate change.

Convexity measures differ among some dealers due to scaling

When calculating the convexity measure, some dealers use ***(2)*** in the denominator and some do not.

For example, our previous definition of convexity was:

$$\text{Approximate convexity} = \frac{V_+ + V_- - 2\,V_0}{\boldsymbol{(2)}\,V_0(\Delta y)^2}$$

..and the application of convexity shown was:

Convexity measure x (yield change)^2 x 100

However, it is also possible to show convexity as:

$$\text{Approximate convexity} = \frac{V_+ + V_- - 2\,V_0}{V_0(\Delta y)^2}$$

With convexity adjustment to percentage price change =

(Convexity measure/2) x (yield change)^2 x 100

Either approach is fine as long as the correct convexity adjustment formula is used. This is more important than the convexity measure in isolation.

Explain the difference between modified convexity and effective convexity.

Modified convexity (convexity) assumes that the cash flows do not change when the yield changes. **Effective convexity** assumes that cash flows do change when yields change. This terminology is the same as with duration.

Example: Suppose that Raquel Carella buys a 9%, 30-year bond for \$1000. Suppose Raquel's bond has a put option that enables her to sell the bond back to the firm at par in the event that interest rates rise. Determine the effective convexity for Raquel's bond.

Price	Current	Put Price	-100bp	+100bp	
	1000	1000	1113.17	905.35	
Modified Convexity					
Effective Convexity					???

$$\frac{V_{+} + V_{-} - 2\,V_0}{\boldsymbol{(2)}\ V_0(\Delta y)^2}$$

Convexity measure x (yield change)^2 x 100

Modified convexity = $\dfrac{905.35 + 1113.17 - 2(1000)}{\boldsymbol{(2)}\ 1000(.01)^2} = 46.30$

Effective convexity = $\dfrac{1000 + 1113.17 - 2(1000)}{\boldsymbol{(2)}\ 1000(.01)^2} = 282.93$

In this case, the put option clearly increases the positive convexity. In the absence of using the effective convexity computation, the modified computation would distort the true value of this bond with its embedded put option.

Price value of a basis point

The **price value of a basis point** (PVBP) also known as **dollar value of an 01** (VD01) is the absolute value of the change in price of a bond for a 1 basis point change in yield.

PVBP = | initial price – price if yield is changed by 1 BP |

It's assumed that price changes will be approximately the same for a small increase or decrease in basis points. In fact, the PVBP is a special case of dollar-duration, which looks at a 100BP change. If duration is 10.66 then the percentage price change for PVBP equal 1-.66 * (0.0001) * 100 = 0.1066%

If the initial price is 134.6722, the dollar price change using duration is
0.1066% * 134.6722 = $0.1435

Difference between duration and the price value of a basis point:
Duration represents the percentage price change in a bond given a 100BP change in market interest rates. PVBP represents the dollar price change in a bond given a 1 BP change in market interest rates.

Explain the importance of yield volatility in measuring the exposure of a bond position to interest rate risk.

Yield Volatility

As we've seen, time to maturity and a bond's coupon rate influence its duration and sensitivity to market interest rate changes. However, the bond's duration does not, by itself, provide the complete assessment of a bond's interest rate risk. It is also important to examine a bond's credit risk. For example, a low quality (i.e. junk bond) may have a shorter duration than a U.S. Treasury bond, yet still have greater interest rate risk.

When the yield level is high a change in interest rates does not produce a large change in the initial price. However when the yield level is low a change in market interest rates of the same number of basis points may produce a large change in the initial price.

Yield volatility or **interest rate volatility** examines the relative volatility of rates. As the expected yield volatility increases, interest rate risk increases for a given duration and current value of a position. A **value-at-risk framework** attempts to bring together the price sensitivity of a bond position to rate changes and yield volatility. This is discussed in greater depth at Level II.

Problems: Fabozzi, Chapter 7

1. An agency bond has a stated duration of 8.53 years and a convexity of 124.77. This suggests that:
   A. if market yields increase significantly (e.g., rates increase by 250 basis points), the price of the bond will fall by less than the amount indicated by duration alone.
   B. if market yields increase significantly, the price of the bond will fall by more than the amount indicated by duration alone.
   C. if market yields decrease significantly (e.g. by 250 basis points), the price of the bond will increase by less than the amount indicated by the convexity measure alone.
   D. if market yields decrease significantly, the price of the bond will increase by less than the amount indicated by duration alone.

2. Given the following information, compute the approximate duration.

Current Bond Price (yield at 8%)	Bond price with yield at 8.1%	Bond Price with yield at 7.9%
108.47	105.86	111.45

   A. 12.32
   B. 25.77
   C. 16.84
   D. 8.44

3. Alissa Remington had a 9%, 6-year bond with a YTM of 10.37%. Alissa's bond has a duration of 16.74 and a convexity of 37.88. She is concerned about market interest rising in the near future. How much will her bond price change if market interest rates increase by 50 BP?

   A. 16.74%
   B. -33.48%
   C. -16.74%
   D. -8.37%

4. Which of the following statements regarding duration and a bond's price volatility is (are) correct?
   I. Duration is a linear estimate of a bond's price change given an expected change in market interest rates
   II. Duration actually underestimates a bond's price increase and decrease given an expected change in market interest rates
   III. The combined effect of a bond's duration and convexity will be greater than a bond's expected change related to duration alone.

IV. Convexity is an attempt to mitigate the error included with the duration measure

A. I and II only
B. II and III only
C. I and IV only
D. III and IV only

5. Ally McByle owns a 6%, 11-year bond with a Macaulay duration of 9 years, providing a YTM of 8%. Assume that the bond pays semiannually and is expected to have a yield decline by 65 basis points. What is the expected percentage price change in Ally's bond?

A. –6.873%
B. 5.625%
C. 9.650%
D. 5.850%

6. Helen Gambler purchased a 7%, 12-year bond at a price of 115.37. The promised YTM at the date of purchase is 6.2%, though Helen believes that market interests will change by up to 20 basis points soon. Given the information provided below, compute the approximate convexity on Helen's bond:

Current Bond Price	Bond Price with 20BP increase	Bond Price with 20BP decrease
106.702	104.973	108.468

A. 43.35
B. 27.66
C. 18.84
D. 66.93

7. Which of the following statements regarding interest rate volatility and bond pricing is (are) <u>not</u> true?
   I. For a given change in interest rates, longer-term bonds have larger price changes
   II. Price volatility increases at an increasing rate as term to maturity increases
   III. Price movements resulting from equal absolute increase or decreases in yield are not symmetrical. Decreases in interest rates result in increases in bond prices by more than an increase in interest rates on the same amount lowers prices.
   IV. Higher coupon issues demonstrate smaller percentage price fluctuation for a given change in interest rates; thus suggesting that bond price volatility is inversely related to coupon.

A. I and II only
B. II only

C. III only
D. III and IV only

8. From the table of bonds below, which bond is most likely to experience the greatest percentage yield increase, given a 2% reduction in market interest rates?

Name	A	B	C	D
Coupon	6%	9%	6%	9%
Term	5 Year	5 Year	20 Year	20 Year

9. Which of the following statements regarding convexity is (are) incorrect?
   I. Callable bonds have greater convexity than bonds without embedded options
   II. Putable bonds reduce the convexity and are therefore desirable in a portfolio
   III. Convexity measures the rate of change not explained by duration
   IV. Convexity becomes more significant with greater changes in interest rates

A. I only
B. I and II only
C. II and III only
D. IV only

10. Karen Zinman purchased a 7-year, 9% option-free bond for par. Her bond has a modified duration of 5.1. Given an anticipated change in market interest rates of 150 basis points, which of the following scenarios represents the changes estimated by duration compared to the actual results?

	Scenario	**A**	**B**	**C**	**D**
	Market Interest rates				
Duration Estimate (percentage price change)	+ 150 BP	-7.65%	+7.5%	-7.65%	-7.65%
Duration Estimated (percentage price change)	–150BP	+7.65%	+7.65%	+7.65%	+7.5%
Actual percentage price	+150 BP	-8.05%	-7.31%	-7.31%	-7.31%

change					
Actual percentage price change	-150 BP	+7.31%	+8.05%	+8.05%	+8.05%

11. Sally Plains purchased a 9%, 30-year bond in which she paid $1000. Her bond has a put option that enables her to sell the bond back to the firm at par in the event that interest rates rise. Determine the effective duration for Sally's bond.

Price	Current	Put Price	-100bp	+100bp	
	1000	1000	1113.17	905.35	
Modified Duration					
Effective Duration					???

A. 10.39
B. 21.87
C. 7.88
D. 5.66

12. Drew Bleedsowat purchased a 9%, 30-year bond in which he paid $1000. His bond has a call option that enables the issuer to buy back the bond in the event that interest rates rise. Determine the effective convexity for Drew's bond.

Price	Current	Call Price	-100bp	+100bp	
	1000	1050	1113.17	905.35	
Modified Convexity					46.30
Effective Convexity					???

A. –111.63
B. 13.71
C. 93.48
D. 46.30

ANSWERS: Fabozzi, Chapter 7

1. **A**. Positive convexity suggests if market yields increase significantly (e.g., rates increase by 250 basis points), the price of the bond will fall by less than the amount indicated by duration alone.

2. **B**. **Solution:** Approximate duration $= \dfrac{V_- - V_+}{2\,V_o(\Delta y)}$

   $$\frac{111.45 - 105.86}{2 * 108.47 * (.001)} = 25.77$$

3. **C.** This question has plenty of extraneous information. The coupon, term, YTM and convexity are all traps! This is a simple question requiring a simple computation. Take the duration of 16.74 and multiply by –0.5%. Her bond will drop by 8.37%. Note that duration is a linear estimate. If interest rates had been expected to decline by 0.5%, the bond would be expected to rise by 8.37%. In actuality, the same interest rate change does not affect price increases and decreases equally. The notion of convexity causes interest rate declines to cause bonds to rise by more than the same interest rate increase will cause bonds to fall.

4. **C**. Duration is a linear estimate and the application of convexity is an attempt to remedy the errors related to duration. Duration underestimates the bond price increase when market interest rates decline and overestimates the bond price decline when market interest rates rise. Convexity, which can be either positive or negative, may add or reduce the effective change suggested by duration alone.

5. **B**. This question has two traps: both extra information and the "wrong" duration are provided. In this situation, coupon and term to maturity are irrelevant items. Moreover, this question provides *Macaulay* duration, not *modified* duration. If no distinction is made, candidates can safely assume that modified duration is being provided. However, since Macaulay duration, and not modified duration is provided, an adjustment needs to be made prior to determining the influence of interest rate changes on Ally's bond's price. The required adjustment changing Macaulay duration to modified duration is shown below:

   $$D_{mod} = 9/(1 + .08/2)$$

   $$9/1.04 = 8.65 \text{ years}$$

Given a modified duration of 8.65 years, the problem simplifies to the following formula:

$$\frac{\Delta P}{P} \times 100 = -D_{mod} * \Delta y$$

Since the bond's YTM is expected to decline 65 basis points, the expected change in bond price should be:

$$\%\Delta P = -8.65 \times \frac{-65}{100} \qquad = (-8.65) \times (-.65) = 5.625\%$$

6. **A**. The formula for computing the approximate convexity is :

   $$\text{Approximate convexity} = \frac{V_+ + V_- - 2\ V_0}{2\ V_0(\Delta y)^2}$$

   In this particular case, given a 20 basis point change, the terms of the formula equal:
   $V_0 = 106.702$
   $V_+ = 104.973$
   $V_- = 108.468$

   $$\text{Convexity measure} = \frac{104.973 + 108.468 - 2(106.702)}{2(106.702)(0.002)^2} = 43.35$$

7. **B.** Price volatility increases at a *diminishing* rate (not increasing rate) as term to maturity increases. The remaining statements are all true.

8. **C.** The greatest percentage change occurs with the bond having the lowest coupon and the longest maturity. Among the bonds in this table, bond "C" has the lowest coupon (6%) and the longest maturity (20 years). The bond with the 9% coupon and 5 year maturity would have the smallest percentage bond price change, given a change in market interest rates. Note: the 9%, 20-year bond appears to have the greatest price change when interest rates drop from 6% to 4%, since its price rises by $33.72 compared to a $27.36 price change for the 6%, 20-year bond. However, the percentage price increase from 134.67 to 168.39 is only 25.04%. This is a smaller percentage price change compared to the 6%, 20-year bond for the same change in market interest rates (e.g. market interest rates change from 6% to 4%. The percentage price increase for the 6%, 20-year bond is 27.36% (100 to 127.36 provides a percentage price change of 27.36%), which is more than any other bond listed in this table. Remember, it is the percentage price change and NOT the absolute dollar amount change that is important.

Yield	**6%--5yr**	**9%--5yr**	**6%--20yr**	**9%--20yr**
4%	108.98	122.46	127.36	168.39
6%	100	112.80	100.00	134.67
8%	91.89	104.06	80.21	109.90

9. **B.** Callable bonds have negative convexity and are perceived to be unattractive to investors, while putable bonds have increased convexity and are considered desirable. Callable bonds take away the potential for greater returns when interest rates fall, while putable bonds provide a floor or protection from significant losses when interest rates rise.

10. **C**. No calculations are necessary! The astute student will realize that the linear approximation known as duration must show the same increase and decrease changes. Since scenarios "B" and "D" show different duration estimates for 150BP increase and decrease (i.e. 7.5% or 7.65%), these two choices are immediately eliminated. Scenario "B" also has the wrong sign with respect to the market interest rate increase (it was positive and should be negative). Scenario "A" is eliminated since it shows an actual percentage price decrease that is greater than the actual percentage price increase. Given an option-free bond and the presumption of positive convexity, equal interest rate increases and decreases cause the bond to rise by a greater percentage than fall. Scenario "C" is the only situation in which duration increases and decreases are the same (with the correct signs) and which has the correct ordering of actual percentage increases and decreases (i.e. the actual percentage bond price increase is greater than the actual percentage price decrease). Therefore, even with no computations, scenario "C" must be correct. Note: the computation of duration equals 5.1 * 1.5 = +/- 7.65%. The actual bond price with a 150BP increase is 926.93 and the actual bond price with a 150BP decrease is 1080.54. These computations can be determined by n=14, FV =1000, PMT = 45, I = 3.75 (for interest decrease) or I = 5.25 (for interest rate increase), solve for PV.

11. **D**.
    Effective duration = (1113.17 – 1000)/(2*1000*.01) = 5.66
    Modified duration = (1113.17 – 905.35)/(2*1000*.01) = 10.39

    Note that the effective duration computation inserts the put price of $1000 rather than the implied price of $905.35. The put price limits the price depreciation of Sally's bond. In this case her effective duration is lower than the modified duration (e.g. 5.66 versus 10.39). While, the question did not request convexity, this bond would have greater convexity than without the put option.

12. **A.** The formula and computations for both modified convexity and for effective convexity are shown below. However, no computations are necessary! Since this bond has a call option, it has negative convexity. Only answer "A" has a negative answer. Therefore, the astute observer would simply answer this option and continue with the exam.

    Formula for convexity $$\frac{V_{+} + V_{-} - 2\,V_0}{\boldsymbol{(2)}\ V_0(\Delta y)^2}$$

Modified convexity = $\dfrac{905.35 + 1113.17 - 2(1000)}{\boldsymbol{(2)}\ 1000(.01)^2} = 46.30$

Effective convexity = $\dfrac{905.35 + 1050 - 2(1000)}{\boldsymbol{(2)}\ 1000(.01)^2} = -111.63$

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