Solution of the Midterm - MAE143B, Spring 2003

1 Prob. 1

The followings are "true" or "false" questions. Check true or false and show why.

(a) The system below is asymptotically stable.

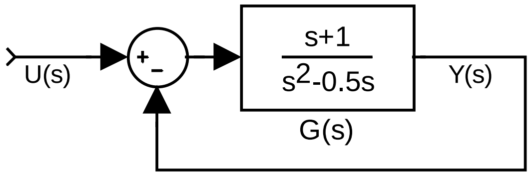

Figure 10: Block Diagram

TRUE FALSE

1.1 Solution of Prob. 1 (a)

The closed loop transfer function is

$$\frac{\frac{s+1}{s(s-0.5)}}{1+\frac{s+1}{s(s-0.5)}} = \frac{s+1}{s(s-0.5)+s+1} = \frac{s+1}{s^2+0.5s+1}.$$

The characteristic equation is

$$s^2 + 0.5s + 1 = 0.$$

The solution of the characteristic equation is

$$s = \frac{-1 \pm \sqrt{1/4-4}}{2}$$

Since $Re(s) = -0.5 < 0$, the closed loop system is stable. Hence the statement is "True".

(b) There exists a proper compensation $C(s)$ which causes the system below to have the closed loop transfer function

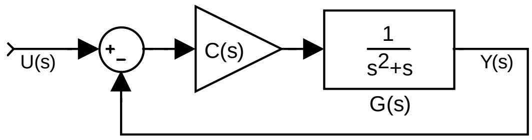

Figure 11: Block Diagram

$$\frac{Y(s)}{U(s)} = H(s) = \frac{s+2}{s^2+2s+2}$$

TRUE FALSE

☐ ☐

1.2 Solution of Prob. 1 (b)

The closed loop transfer function

$$\frac{Y(s)}{U(s)} = H(s) = \frac{\frac{C}{s(s+1)}}{1+\frac{C}{s(s+1)}} = \frac{C}{s(s+1)+C} \;\Rightarrow\; (1-H)\,C = s(s+1)H.$$

Let's denote

$$\mathbf{H} \triangleq \frac{H_n}{H_d}.$$

where

$$H_n \triangleq s+2\ ,\ \ H_d \triangleq s^2+2s+2$$

Then

$$C(s) = \frac{H_n}{H_d - H_n} s(s+1) = \frac{s+2}{s^2+2s+2-s-2} s(s+1) = s+2$$

Hence $C(s) = s+2$ is not proper and the statement is "false".

(c) The steady state response of

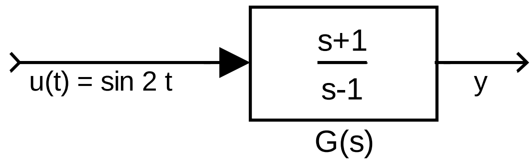

Figure 12: Block Diagram

is

$$y(\infty) = sin(t + 2\pi + 2\tan^{-1}(2))$$

TRUE FALSE

1.3 Solution of Prob. 1 (c)

The input and output relationship is

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s-1} = G(s)$$

Since $G(s)$ has positive pole, the system is unstable. Hence the output $y(t)$ approaches to infinity as time is infinity. But the statement $y(\infty) = sin(t + 2\pi + 2\tan^{-1}(2))$ is bounded, this statement is "false".

2 Prob. 2

Write the ordinary differential equation that relates $y(t)$ to $u(t)$.

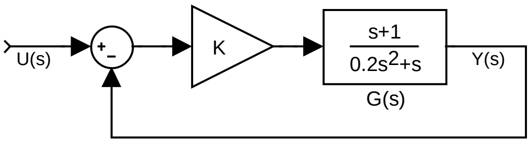

Figure 13: Block Diagram

2.1 Solution of Prob. 2

$$\frac{Y(s)}{U(s)} = \frac{K\frac{s+1}{s(0.2s+1)}}{1+K\frac{s+1}{s(0.2s+1)}} = \frac{K(s+1)}{s(0.2s+1)+K(s+1)}$$

$$[s(0.2s+1)+K(s+1)]\,Y(s) = K(s+1)\,U(s)$$

By inverse Laplace transformation,

$$0.2\ddot{y}(t) + (K+1)\,\dot{y}(t) + Ky(t) = K\dot{u}(t) + Ku(t)$$

3 Prob. 3

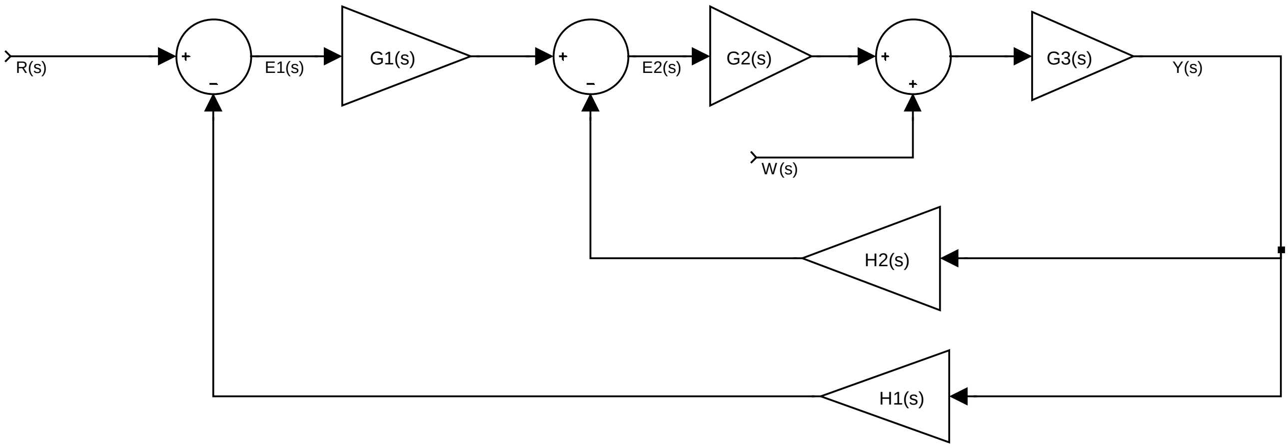

Figure 14: Block Diagram

Consider the feedback control system shown in the above. Derive the following transfer functions [5 points each] : (a) $\frac{Y(s)}{R(s)}$, (b) $\frac{Y(s)}{W(s)}$, (c) $\frac{E2(s)}{R(s)}$, (d) $\frac{E2(s)}{W(s)}$

3.1 Solution of Prob. 3

From the block diagram, we have

$$e_1 = R - H_1 y \quad (2)$$
$$e_2 = G_1 e_1 - H_2 y \quad (3)$$
$$y = (G_2 e_2 + w) G_3 \quad (4)$$

3.1.1 Solutions of (a) and (b)

Key step is to eliminate the signals e_1 and e_2 in (2),(3), and (4).

Substituting (2) to (3) yields

$$e_2 = G_1 (R - H_1 y) - H_2 y \quad (5)$$

Substituting (5) to (4) yields

$$y = [G_2 (G_1 (R - H_1 y) - H_2 y) + w] G_3 \quad (6)$$

Rearranging (6)

$$(1 + G_2 G_3 (H_2 + H_1 G_1)) y = G_1 G_2 G_3 r + G_3 w$$

Hence

$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 + G_2 G_3 (H_2 + H_1 G_1)}$$

and

$$\frac{Y(s)}{W(s)} = \frac{G_3}{1 + G_2 G_3 (H_2 + H_1 G_1)}$$

3.1.2 Solutions of (c) and (d)

Key step is to eliminate the signals y and e_2 in (2),(3), and (4).

Substituting (4) into (5), we have

$$\begin{aligned} e_2 &= G_1R - (G_1H_1 + H_2)y \\ &= G_1R - (G_1H_1 + H_2)(G_2e_2 + w)G_3 \end{aligned}$$

Rearranging the above equation, we have

$$(1 + (G_1H_1 + H_2)G_2G_3)\, e_2 = G_1R - (G_1H_1 + H_2)G_3w$$

Hence

$$\frac{E_2(s)}{R(s)} = \frac{G_1}{(1 + (G_1H_1 + H_2)G_2G_3)}$$

and

$$\frac{E_2(s)}{W(s)} = -\frac{(G_1H_1 + H_2)G_3}{(1 + (G_1H_1 + H_2)G_2G_3)}$$

4 Prob. 4

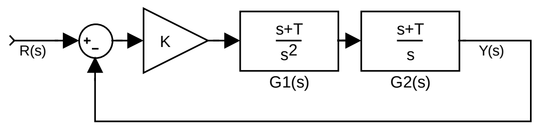

Figure 15: Block Diagram

(a) Find the values of K and T for which the system shown below is stable.

4.1 Solution of Prob. 4(a)

$$\frac{Y(s)}{R(s)} = \frac{G}{1+G} = \frac{k(S+T)^2}{s^3 + K(S+T)^2}$$

The characteristic equation is

$$s^3 + K(S+T)^2 = s^3 + Ks^2 + 2KTs + KT^2 \triangleq s^3 + as^2 + bs + c$$

where

$$a \triangleq K \ , \ b \triangleq 2KT \ , \ c \triangleq KT^2$$

The Routh's array is

$$\begin{array}{lcll} s^3 & : & 1 & b \\ s^2 & : & a & c \\ s^1 & : & \frac{ab-c}{a} & 0 \\ s^0 & : & c & 0 \end{array}$$

For stability, we have

$$a = K > 0 \ , \ c = KT^2 > 0 \ , \ \frac{ab - c}{a} = \frac{K2KT - KT^2}{K} = T(2K - T) > 0$$

which is equivalent to

$$K > 0 \ , \ T(2K - T) > 0$$

or equivalently

$$K > 0 \ , \ sign(T) = sign(2K - T)$$

Let's analyze the second condition further.

$$if\ T > 0,\ then\ 2K - T > 0\ \Rightarrow 2K > T > 0 \quad (7)$$
$$if\ T < 0,\ then\ 2K - T < 0\ \Rightarrow 2K < T < 0 \quad (8)$$

(7) is fine, while (8) contradicts $K > 0$.
Hence the stability condition is

$$2K > T > 0.$$

(b) When $K = \frac{1}{2}$, find the values of $T > 0$ such that all poles lie strictly to the left of the vertical line $s = -\frac{T}{2} \pm j\omega \ , \ 0 \le \omega \le \infty$, if it is possible.

4.2 Solution of Prob. 4(b)

Consider the change of variable

$$p = s + \frac{T}{2}$$

Then the characteristic equation is

$$\begin{aligned} s^3 + K(s+T)^2 &= (p - \frac{T}{2})^3 + K(p + \frac{T}{2})^2 \\ &= (p - \frac{T}{2})(p^2 - pT + \frac{T^2}{4}) + K(p^2 + pT\frac{T^2}{4}) \\ &= p^3 - Tp^2 + T^2p/4 - Tp^2/2 + T^2p/2 - T^3/8 + K(p^2 + pT\frac{T^2}{4}) = 0 \end{aligned}$$

Since $K = \frac{1}{2}$, there is a solution $p = 0$ satisfying the above equation. Hence there is no T such that all poles lie strictly to the left of the vertical line $s = -\frac{T}{2} \pm j\omega \ , \ 0 \le \omega \le \infty$.

5 Prob. 5

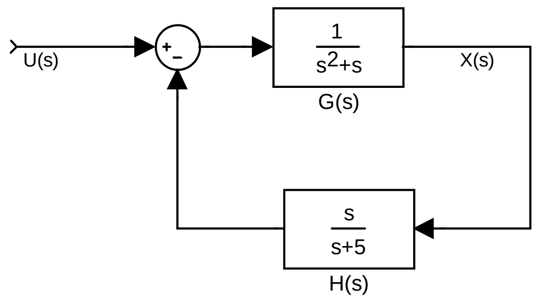

Figure 16: Block Diagram

(a) Find $x(t)$ if $u(t)$ is a unit impulse function.

(b) Explain how you would change $H(s)$ to guarantee that $lim_{t\to\infty} x(t) \to 0$ in the presence of a ramp input $u(t) \triangleq kt$.

5.1 5.(a)

$$X(s) = \frac{G(s)}{1+G(s)H(s)}U(s) = \frac{s+5}{s(s^2+6s+6)} = \frac{s+5}{s(s+3+\sqrt{3})(s+3-\sqrt{3})}$$

From partial fraction formula,

$$\begin{aligned} X(s) &= \frac{s+5}{s(s+3+\sqrt{3})(s+3-\sqrt{3})} \\ &= \frac{A}{s} + \frac{B}{s+3+\sqrt{3}} + \frac{C}{s+3-\sqrt{3}} \end{aligned}$$

where

$$\begin{aligned} A &= sX(s)|_{s=0} = \frac{5}{6} \\ B &= (s+3+\sqrt{3})X(s)|_{s=-3-\sqrt{3}} = \left.\frac{s+5}{s(s+3-\sqrt{3}}\right|_{s=-3-\sqrt{3}} = \frac{2-\sqrt{3}}{2\sqrt{3}(3+\sqrt{3})} \\ C &= (s+3-\sqrt{3})X(s)|_{s=-3+\sqrt{3}} = \left.\frac{s+5}{s(s+3+\sqrt{3}}\right|_{s=-3+\sqrt{3}} = \frac{2+\sqrt{3}}{2\sqrt{3}(-3+\sqrt{3})} \end{aligned}$$

5.2 5.(b)

From the final value theorem,

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\frac{G}{1+GH}U(s) = \lim_{s\to 0} s\frac{1}{s(s+1)+H}U(s)$$

Since $u(t) = kt$,

$$U(s) = \frac{k}{s^2}.$$

Hence

$$x(\infty) = \lim_{s\to 0} s \frac{1}{s(s+1)+H}\frac{k}{s^2} = \lim_{s\to 0} \frac{1}{s(s+1)+H}\frac{k}{s} = \lim_{s\to 0}\frac{k}{sH(s)} = 0$$

which means that $sH(s)|_{s=0} = \infty$. Thus $H(s)$ must have s^2 in the denominator and the roots of $1+GH$ must lie in the LHP.

6 Prob. 6

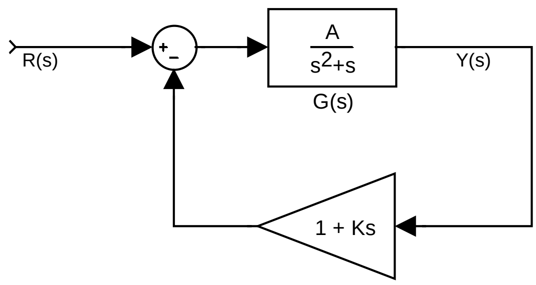

Figure 17: Block Diagram

Consider the feedback control system shown in the above.

(a) Determine A and K to satisfy the following specifications.

- (i) The transfer function $\frac{\mathbf{Y}(s)}{R(s)}$ is stable.
- (ii) maximum overshoot (M_p) for a unit step input of less than 17%
- (iii) 3% settling time (t_s) of less than 3.5 sec.

(b) Determine the system type with respect to the output y and the error($e = r - y$) .

6.1 6.(a)

(i) The transfer function $\frac{\mathbf{Y}(s)}{R(s)}$ is stable.

$$\frac{Y(s)}{R(s)} = \frac{G}{1+GH} = \frac{A}{s(s+1)+A(1+Ks)} = \frac{A}{s^2+(1+AK)s+A} \triangleq \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Hence we have

$$2\zeta\omega_n = 1+AK \ , \ \omega_n^2 = A \tag{9}$$

The Routh's array is

$$\begin{array}{lcll} s^2 & : & 1 & A \\ s^1 & : & 1+AK & 0 \\ s^0 & : & A & \end{array}$$

Hence we have

$$A > 0 \ , \ 1+AK > 0$$

(ii) maximum overshoot (M_p) for a unit step input of less than 17%

$$M_p = 17\% \;\Leftrightarrow e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.17 \;\Rightarrow\; \zeta = 0.5$$

(iii) 3% settling time (t_s) of less than 3.5 sec.

$$e^{-\zeta\omega_n t_s} = 0.03 \;\Rightarrow \zeta\omega_n = -\frac{ln(0.03)}{t_s} = \frac{3.5}{3.5} = 1$$

Since $\zeta = 0.5$, $\omega_n = 2$. From (9), we have

$$A = \omega_n^2 = 4 \;,\; 2\zeta\omega_n = 2 = 1 + AK = 1 + 4K$$

Hence $1 + 4K = 2 \;\Rightarrow\; K = 0.25$.

6.2 6.(b)

The error signal is

$$E(s) = R(s) - Y(s) = \left(1 - \frac{Y(s)}{R(s)}\right) R(s) = \frac{s(s+1+AK)}{s^2 + (1+AK)s + A} R(s)$$

From the final value theorem,

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{s(s+1+AK)}{s^2+(1+AK)s+A} R(s) \\ &= \lim_{s\to 0} s \frac{s(s+1+AK)}{s^2+(1+AK)s+A} \frac{1}{s^2} \\ &= \lim_{s\to 0} \frac{(s+1+AK)}{s^2+(1+AK)s+A} = \frac{1+AK}{A} = \frac{2}{4} = 0.5 \end{aligned}$$

7 Prob. 7

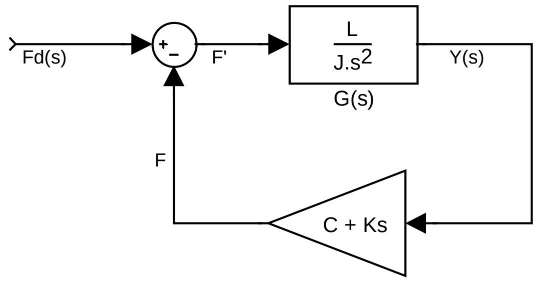

Figure 18: Block Diagram

A rigid spacecraft is controlled by reaction jets which operate in pairs to produce the torque FL . A position plus rate feedback (PD) controller is employed for the controller $H(s)$. The rate gyro gain, K, and the position gain ,C, ae to be determined.

(a) When $J = 1$ and $L = 4$, determine K and C such that (i) and (ii) hold.

(i) The impulse response shows no oscillations.

(ii) The steady state position error, $\theta_e(\infty)$, is less than 0.01 in the presence of an effective bias disturbance of magnitude $F_d(t) = 1$.

(b) Let $C = 1$ and $K = 0.707$. Find a proper location (L) for the jets so that the system damping ratio ζ is 0.707. That is, find L such that $\zeta = 0.707$.

(c) Suggest a feedback controller $H(s)$ which will guarantee $\lim_{t\to\infty} \theta_e(t) = 0$ regardless of the magnitude of the bias disturbance $F_d(t) = F_o$. (Show this result of your design) .

(d) Write a set of state equations for the system in the above block diagram. Define the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$ and the input as $u = F_d$. Find the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and D such that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \ y = \mathbf{C}\mathbf{x} + Du$$

where $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $y \triangleq x_1$.

(e) As a check on your answer in (e), compute the transfer function

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + D$$

and compare this answer with

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

7.1 7.(a)

(i) The impulse response shows no oscillations. ⇒ The characteristic equation must have real roots. Since the transfer function

$$\frac{\theta_e}{F_d} = \frac{G}{1+GH} = \frac{L}{Js^2 + LKs + LC},$$

the characteristic equation is

$$\frac{L}{Js^2 + LKs + LC} = 0,$$

or equivalently

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + \frac{LK}{J}s + \frac{LC}{J} = 0$$

Thus

$$\omega_n^2 = \frac{LC}{J} = 4C \ , \ 2\zeta\omega_n = \frac{LK}{J} = 4K$$

Hence

$$\omega_n = 2\sqrt{C} \ , \ \zeta = \frac{4K}{2\omega_n} = \frac{K}{\sqrt{C}}$$

For real roots,

$$\zeta \geq 1 \Rightarrow \frac{K}{\sqrt{C}} \geq 1 \tag{10}$$

(ii) The steady state position error, $\theta_e(\infty)$, is less than 0.01 in the presence of an effective bias disturbance of magnitude $F_d(t) = 1$.

Since the transfer function

$$\frac{\theta_e}{F_d} = \frac{G}{1+GH} = \frac{L}{Js^2 + LKs + LC},$$

and using $F_d(s) = \frac{1}{s}$, we have

$$\theta_e(s) = \frac{L}{s(Js^2 + LKs + LC)}$$

From final value theorem,

$$\theta_e(\infty) = \lim_{t\to\infty} s\theta_e(s) = \lim_{t\to\infty} s\frac{L}{s(Js^2 + LKs + LC)} = \frac{1}{C}$$

From the requirements $\theta_e(\infty) < 0.01$, we have

$$\theta_e(\infty) = \frac{1}{C} < 0.01 \ \Rightarrow C > 100. \tag{11}$$

From (10) and (11), the conditions (i) and (ii) are equivalent to

$$\frac{K}{\sqrt{C}} \geq 1 \ , \ \theta_e(\infty) = \frac{1}{C} < 0.01$$

One acceptible choise is

$$C = 200 \ , \ K = 300$$

7.2 7.(b)

$$\zeta = 0.707 = \frac{LK/J}{2\sqrt{LC/J}}$$

$$L = 4C = 4$$

7.3 7.(c)

$$\theta(s) = \frac{L/(Js^2)}{1 + H(s)L/(Js^2)}\frac{F_o}{s}$$

$$\theta(\infty) = [s\theta(s)]_{s=0} = \left.\frac{LF_o}{Js^2 + LH(s)}\right|_{s=0} = \left.\frac{LF_o}{LH(s)}\right|_{s=0} = \frac{F_o}{H(0)} = 0 \text{ if } H(0) = \infty.$$

Hence choose any $H(s)$ which has a pole at the origin, such as

$$H(s) = \frac{Ks + C}{s}.$$

7.4 7.(d)

$$\frac{\theta(s)}{F_d(s)} = \frac{L}{Js^2 + LKs + LC}$$

Cross multiplying and taking the inverse Laplace transformation yield

$$J\ddot{\theta} + LK\dot{\theta} + LC = LF_d(t)$$

Choose the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{LK}{J}x_2 - \frac{LC}{J}x_1 + \frac{L}{J}F_d \end{aligned}$$

Hence we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \; y = \mathbf{C}\mathbf{x} + Du$$

where $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $y \triangleq x_1$, and

$$\mathbf{A} \triangleq \begin{bmatrix} 0 & 1 \\ -\frac{LC}{J} & -\frac{LK}{J} \end{bmatrix}, \; \mathbf{B} \triangleq \begin{bmatrix} 0 \\ \frac{L}{J} \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0$$

7.5 7.(e)

$$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+D=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ \frac{LC}{J} & s+\frac{LK}{J} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{L}{J} \end{bmatrix}$$

Since

$$\begin{bmatrix} s & -1 \\ \frac{LC}{J} & s+\frac{LK}{J} \end{bmatrix}^{-1}=\frac{1}{s(s+\frac{LK}{J})+\frac{LC}{J}}\begin{bmatrix} s+\frac{LK}{J} & 1 \\ -\frac{LC}{J} & s \end{bmatrix},$$

we have

$$\begin{aligned}\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+D &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ \frac{LC}{J} & s+\frac{LK}{J} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{L}{J} \end{bmatrix} \\ &= \frac{1}{s(s+\frac{LK}{J})+\frac{LC}{J}}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+\frac{LK}{J} & 1 \\ -\frac{LC}{J} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{L}{J} \end{bmatrix} \\ &= \frac{\frac{L}{J}}{s(s+\frac{LK}{J})+\frac{LC}{J}}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix}=\frac{L}{Js^2+LKs+LC}\end{aligned}$$

which agrees with

$$\frac{Y(s)}{U(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{L}{Js^2+LKs+LC}.$$