Exercise 18.1 Use the adjoint method to compute the inverse of

c) $\begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ $A^{-1} = (1/|A|)*(A^c)^T = (-1/12)*\begin{bmatrix} -12 & 12 & 20 & 24 \\ 0 & -6 & -2 & -3 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

The steps to arrive at this are:

Step (1) $|A| = -12$

Since the matrix is in triangular form, you can use the diagonal to derive the determinate which is: (1)*(2)*(3)*(-2)= -12

Step (2)

$$(A^c) = (-1)^{(1+1)}\begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(1+2)}\begin{vmatrix} 0 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(1+3)}\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(1+4)}\begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

$$(-1)^{(2+1)}\begin{vmatrix} 2 & 4 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(2+2)}\begin{vmatrix} 1 & 4 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(2+3)}\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(2+4)}\begin{vmatrix} 1 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

$$(-1)^{(3+1)}\begin{vmatrix} 2 & 4 & -3 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(3+2)}\begin{vmatrix} 1 & 4 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(3+3)}\begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{vmatrix}, (-1)^{(3+4)}\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$(-1)^{(4+1)}\begin{vmatrix} 2 & 4 & -3 \\ 2 & -1 & 1 \\ 0 & 3 & 0 \end{vmatrix}, (-1)^{(4+2)}\begin{vmatrix} 1 & 4 & -3 \\ 0 & -1 & 1 \\ 0 & 3 & 0 \end{vmatrix}, (-1)^{(4+3)}\begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix}, (-1)^{(4+4)}\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{vmatrix}$$

Calculate the answers for each and plug it into the matrix:

$(A^c) = \begin{bmatrix} -12 & 0 & 0 & 0 \\ 12 & -6 & 0 & 0 \\ 20 & -2 & -4 & 0 \\ 24 & -3 & 0 & 6 \end{bmatrix}$ **Step (3)** $(A^c)^T \begin{bmatrix} -12 & 12 & 20 & 24 \\ 0 & -6 & -2 & -3 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Final Step (4) $A^{-1} = (1/|A|)*(A^c)^T = (-1/12)*\begin{bmatrix} -12 & 12 & 20 & 24 \\ 0 & -6 & -2 & -3 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Exercise 18.2 **Prove $|A^{-1}| = \frac{1}{|A|}$ if A is nonsingular.**

A is invertible (by Theorem 18.2) thus A^{-1} exists and $A^{-1}(A) = 1$.

Thus $|A|*|A^{-1}| = |AA^{-1}| = |I| = 1$ Thus $|A^{-1}| = \frac{1}{|A|}$.

Exercise 18.4 **Find $(A^{-1})_{23}$ for each matrix A:** $A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

First - solve for the determinant:

Since the matrix is in a triangular form: $(1)*(-4)*(2) = -8 \quad \therefore |A| = -8$

Second – solve for A_{32}^{C} (use 32 instead of 23 because it will be transposed)

$$(-1)^{(3+2)} \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = (-1)^{(5)} [\,(-3+0)] = 3$$

Third – Plug results into equation: $\frac{1}{|A|} * (A^{C})_{23}^{T} = (\frac{-1}{8})3$ or $\frac{-3}{8}$

Exercise 18.5 **For a positive integer n, we define A^{-n} to be $(A^{-1})^{n}$.**

Find A^{-3} for $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

First - solve for the determinant: $|A| = [\,(1*2) - (3*(-1))] = 5$

Second – solve for (A^{C}): $\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$

Third – solve for $(A^{C})^{T}$: $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

So $A^{-1} = \frac{1}{|A|} * (A^{C})^{T} = (\frac{1}{5})\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$, and $A^{-2} = (\frac{1}{25})\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = (\frac{1}{25})\begin{bmatrix} 1 & -9 \\ 3 & -2 \end{bmatrix}$

And $A^{-3} = (\frac{1}{125})\begin{bmatrix} 1 & -9 \\ 3 & -2 \end{bmatrix} * \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = (\frac{1}{125})\begin{bmatrix} -7 & -12 \\ 4 & -11 \end{bmatrix}$

Exercise 18.7 Solve with Cramer's Rule:

a.) x + 5z = 1, 3x + 2y = -8, -x + 2y – 19z= -11 by Cramer's Rule

$$\vec{b}\begin{bmatrix} 1 \\ -8 \\ -11 \end{bmatrix} \quad A=\begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & 0 \\ -1 & 2 & -19 \end{bmatrix} \quad |A| = 1\begin{vmatrix} 2 & 0 \\ 2 & -19 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = (-38 + 40) = 2$$

$$A_1 = \begin{bmatrix} 1 & 0 & 5 \\ -8 & 2 & 0 \\ -11 & 2 & -19 \end{bmatrix} \text{ and } |A_1| = 1\begin{vmatrix} 2 & 0 \\ 2 & -19 \end{vmatrix} + 5\begin{vmatrix} -8 & 2 \\ -11 & 2 \end{vmatrix} = (-38 + 30) = -8$$

$$A_2 = \begin{bmatrix} 1 & 1 & 5 \\ 3 & -8 & 0 \\ -1 & -11 & -19 \end{bmatrix} \text{ and } |A_2| = 5\begin{vmatrix} 3 & -8 \\ -1 & -11 \end{vmatrix} + (-19)\begin{vmatrix} 1 & 1 \\ 3 & -8 \end{vmatrix} = (-205 + 209) = 4$$

$$A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -8 \\ -1 & 2 & -11 \end{bmatrix} \text{ and } |A_3| = 1\begin{vmatrix} 2 & -8 \\ 2 & -11 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = (-6 + 8) = 2$$

$$X = \frac{-8}{2} = -4, \quad Y = \frac{4}{2} = 2, \quad Z = \frac{2}{2} = 1$$

Check: -4 + 5(1) = 1 ok, 3(-4) + 2(2) = -8 ok, -(-4) + 2(2) – 19(1) = -11 ok

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b.) x + y = 5, x – y = 1 by Cramer's Rule

$$\vec{b}\begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad |A| = -2$$

$$A_1 = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } |A_1| = -6$$

$$A_2 = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} \text{ and } |A_2| = -4$$

$$X = \frac{-6}{-2} = 3, \quad Y = \frac{-4}{-2} = 2$$

Check: 3 + 2 = 5 ok , 3 –2 = 1 ok