§24 1 (b), 2 (b), 3, 4, 6

Exercise 24.1) In each case, determine whether the system A $\vec{X} = \vec{b}$ is solvable and, if so, how many arbitrary variables will appear in the solution set.

b.) $$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ 1 & 1 & 2 & 1 & 3 \\ 2 & 0 & 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 3/2 & 1 & 5/2 \\ 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A^{+}_{row}, \text{ therefore } r(A^{+})=2$$

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 3/2 & 1 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{row}, \text{ therefore } r(A)=2$$

- ✓ Since: r(A)= 2 = r(A^{+}) it <u>IS SOLVABLE,</u>
- ✓ Arbitrary variables : dimN(A)= n - r(A) = 4 – 2 = <u>2</u>

Exercise 24.2) In each case, solve the system A $\vec{X} = \vec{b}$, leaving your solution in the form of Theorem 24.4.

b.) $$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 1 & 1 & 1 & 2 \\ 1 & 4 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 6 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -\frac{5}{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{10}{3} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{6} \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} 10/3 \\ -1/2 \\ -5/6 \end{bmatrix}$$

Exercise 24.3) Suppose $\vec{X}$ and $\vec{Y}$ are solutions of the non-homogeneous system $A\vec{X} = \vec{b}$. Show $\vec{Z} = w_1\vec{X} + w_2\vec{Y}$ is a solution of $A\vec{X} = \vec{b}$ if and only if $w_1 + w_2 = 1$ (convex property).

(ANSWER IN BACK OF THE TEXTBOOK)

($\Rightarrow$) Suppose $\vec{Z} = w_1\vec{X} + w_2\vec{Y}$ is a solution of $A\vec{V} = \vec{b}$.

Then $\vec{b} = A\vec{Z} = A(w_1\vec{X} + w_2\vec{Z}) = w_1 A\vec{X} + w_2 A\vec{Z} = w_1\vec{b} + w_2\vec{b}$.

Thus, $\vec{b} = (w_1 + w_2)\vec{b} = \vec{0}$

Since $\vec{b} \neq \vec{0}$ then $w_1 + w_2 = 1$.

($\Leftarrow$) Suppose $w_1 + w_2 = 1$

Then $A\vec{Z} = A(w_1\vec{X} + w_2\vec{Y}) = w_1 A\vec{X} + w_2 A\vec{Y} = w_1\vec{b} + w_2\vec{b} = (w_1 + w_2)\vec{b} = 1\vec{b} = \vec{b}$.

Thus, $\vec{Z}$ is a solution.

Exercise 24.4) If $\mathbb{S} \subset \Re^m$ and $\vec{X} \in \Re^m$, we will say that $\vec{X}$ is orthogonal to $\mathbb{S}$ (notation: $\vec{X} \perp \mathbb{S}$) if $\vec{X} \bullet \vec{Y} = 0$ for every $\vec{Y}$ in $\mathbb{S}$. Further, for $\mathcal{T} \subset \Re^m$ we say $\mathcal{T} \perp \mathbb{S}$ if $\vec{X} \perp \mathbb{S}$ for all $\vec{X}$ in $\mathcal{T}$. For all parts below, assume A is an $m \times n$ matrix.

Before we address the questions, let look at these notes:

E vector space E_1, E_2, subspace of E

$E_1 \perp E_2$ is orthogonal if $\vec{X}_1 \perp \vec{X}_2 = 0$ for all $X_1 \in E_1$ and $X_2 \in E_2$ $E_1 \cap E_2 = \{0\}$

$\vec{X} \in E_1 \cap E_2$ $\vec{X} = (X_1, \dots, X_2)$ $E_1 \perp E_2 \Rightarrow \vec{X}_1 \perp \vec{X}_2 = 0$

$\sum_{k=1}^{n} X_k^2 = 0$ $X_k = 0$ for k=1,...,n

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a.) Prove that if $\vec{Y} \perp C(A)$, then $\vec{Y} \in N(A^T)$.
Answer: $A^T \vec{Y} = \vec{0}$ equivalent to $\forall_I\ A_{ei} \bullet \vec{Y} = 0$

b.) Prove $C(A) \perp N(A^T)$.
Answer: $\vec{X} \in C(A)$ and $\vec{Y} \in N(A^T)$ (see answer (a) as the 1st half of this proof)
$\vec{X} \bullet \vec{Y} = 0$

c.) Use Exercise 22.13 to show that $\dim N(A^T) + \dim C(A) = m$.
Answer: Since $\dim C(A) = \dim R(A)$ then $\dim C(A^T)$
Therefore $= \dim N(A^T) + \dim C(A) = m$

d.) If B_1 is a basis for C(A) and B_2 is a basis for $N(A^T)$, prove $B_1 \cup B_2$ is a basis for $\Re^m$.

Answer: Show that $B_1 \cup B_2$ is a basis for $\Re^m$ will be $\Re^m$ but also need to be Linearly Independent. (Look to answer (c) for part of the explanation). $N(A^T) \perp C(A)$

e.) Prove that if $\vec{X} \perp N(A^T)$, then $\vec{X} \in C(A)$.

Answer: B_1 is a basis for C(A) $B_1 = \{x_1, \ldots, x_n\}$

B_2 is a basis for $N(A^T)$ $B_2 = \{y_1, \ldots, y_n\}$

$\vec{X} = \alpha x_1, \ldots, \alpha x_n$ $B_1 Y_1 + \ldots + B_k Y_k \neq 0$