Complete set of Eigenvectors. $\lambda_1,...,\lambda_k$ **Eigenvalues.** **A is an n x n matrix.**

$B_1,...,B_k$ bases $N(A-\lambda_j I)$ B= $B_1 \cup ... \cup B_k \subset R^n$ Sp $B \subset R^n$

If Span $B = R^n$**, then we call B the full set of Eigenvectors. B had n vectors.**

Example where we have a full set of Eigenvectors:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad C(A) = |A - \lambda I| \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 3$ **So** $\lambda = 2$ **has a multiplicity of one and** $\lambda = 3$ **has a multiplicity of 2.**

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \\ 3z \end{bmatrix} \qquad \text{diag}\{2,3,3\}$$

$$N(A-2I) = \text{Sp}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} = B_1 \qquad \text{also} \qquad N(A-3I) = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} = B_2$$

$\text{Sp}\{B_1 \cup B_2\} = R^n$

THEOREM

A has a full set of eigenvectors If and Only If (IFF) A is similar to a diagonal matrix.

PROOF (⇨) (the 1st direction)

A has a full set of eigenvectors. $V_1,...,V_n$

We have to find a non-singular matrix P such that $P^{-1}AP$ **is a diagonal matrix.**

Define: $P = [V_1,...,V_n]$ **(P is non-singular)**

$I = P^{-1}P = P^{-1}[V_1,...,V_n] = [P^{-1}V_1,...,P^{-1}V_n]$ **for K= , ...,n** $P^{-1}V_k = e_k$

$P^{-1}AP = P^{-1}A[V_1,...,V_n] = P^{-1}[\lambda_1 V_1, \lambda_2 V_2 ..., \lambda_n V_n] = [P^{-1}\lambda_1 V_1, P^{-1}\lambda_2 V_2 ..., P^{-1}\lambda_n V_n] =$
$[\lambda_1 P^{-1}V_1,...,\lambda_n P^{-1}V_n] = [\lambda_1 e_1,...,\lambda_n e_n] = \text{diag}\{\lambda_1,...,\lambda_n\}$

PROOF (←) (the other direction)

PAP^{-1} = diag $\{\lambda_1,...,\lambda_n\}$ = B A and B have the same eigenvalues. Also, the geometric multiplicity is the same for each eigenvalue. dim(N(A- λI)) = dim(N(B- λI)). The complete set of eigenvalues has n elements. (A full set). Full set spans full space.

EXAMPLE:

$$A=\begin{bmatrix}2&1&0\\0&1&0\\-1&-1&3\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2-\lambda&1&0\\0&1-\lambda&0\\-1&-1&3-\lambda\end{bmatrix}=0$$

$(3-\lambda)*(2-\lambda)*(1-\lambda)=0$ Eigenvalues are $\lambda=1,\ \lambda=2, \lambda=3$

when $\lambda = 1$ then $$\begin{bmatrix}2&1&0\\0&1&0\\-1&-1&3\end{bmatrix}-1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&0\\0&0&0\\-1&-1&2\end{bmatrix}$$

$$\begin{bmatrix}1&1&0\\0&0&0\\-1&-1&2\end{bmatrix}*\begin{bmatrix}x\\y\\z\end{bmatrix}=\vec{0} \qquad \begin{cases}x+y=0\Rightarrow x=-y\\-x-y+2z=0\\z=0\end{cases} \qquad V_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

when $\lambda = 2$ then $$\begin{bmatrix}2&1&0\\0&1&0\\-1&-1&3\end{bmatrix}-2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&-1&0\\-1&-1&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&-1&0\\-1&-1&1\end{bmatrix}*\begin{bmatrix}x\\y\\z\end{bmatrix}=\vec{0} \qquad \begin{cases}y=0\\-y=0\\-x-y+z=0\end{cases} \qquad V_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

when $\lambda = 3$ then $$\begin{bmatrix}2&1&0\\0&1&0\\-1&-1&3\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&1&0\\0&-2&0\\-1&-1&0\end{bmatrix}$$

$$\begin{bmatrix}-1&1&0\\0&-2&0\\-1&-1&0\end{bmatrix}*\begin{bmatrix}x\\y\\z\end{bmatrix}=\vec{0} \qquad \begin{cases}-x+y=0\\-2y=0\\-x-y=0\end{cases} \qquad V_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Therefore P equals these vectors $\begin{bmatrix}1&1&0\\-1&0&0\\0&1&1\end{bmatrix}$ and thus $P^{-1}AP=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$