A Visual Demonstration of the Fundamental Theorem of Calculus

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This short paper presents an original, **simple demonstration of the Fundamental Theorem of Calculus**. This theorem asserts that finding slopes of tangent lines (i.e. differentiation) and finding areas under curves (i.e. integration) are inverse operations save a constant. Can we prove this? Yes. Most textbooks, in fact, present an elegant, simple proof for this theorem. However, the standard proofs usually omit references to **curves**, **slopes**, and **areas**--a weakness that can make the proof unnecessarily abstract and difficult to understand.

In the next few pages we demonstrate visually the inverse relationship between differentiation and integration; we will see that finding areas under a curve is really the "opposite" of finding slopes of tangent-lines. Before beginning, I assume that the reader is familiar with Riemann's Sums and thus the correspondence between the function $\lim_{\Delta x \to 0} \sum_{k=1}^{n} h(x_k)\Delta x$ and the area under a curve.

For our demonstration, we will need some arbitrary function. For this, we will use the function *f(x)* = x^2. To the right is a graph of that function:

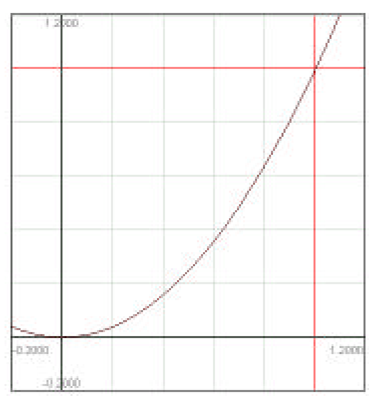

Figure 1. ***f(x)***

If we so wanted, we could construct a graph of the slope of every line tangent to *f(x)* at *x*. Looking at the above graph we see that a line tangent to *f(x)* at $x = 0.2$ would have a slope of 0.4; a line tangent at $x = 0.6$ would have a slope of 1.2; and a line tangent at $x = 1$ would have a slope of 2. Plotting these and similar points we would end up drawing a graph that looks like this:

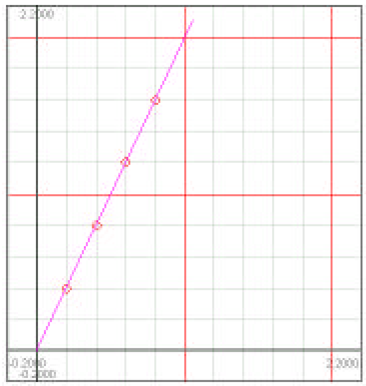

Figure 2. ***f'(x)***

This graph is actually the plot of the function $f'(x) = 2x$, or what we usually call the derivative of *f(x)*. So far we've discussed **differentiation**; let's move on to **integration**.

Imagine we wanted to reverse the above procedure. Imagine we are given an arbitrary function *g(x)* and asked to draw the graph of another function *G(x)* with the following condition: the slope of every line tangent to *G(x)* at some *x* is determined by *g(x)*. How would we go about constructing *G(x)*?

Let's use a simple example. We begin with the function $g(x) = 2x$. We would like to draw the graph of another function *G(x)* whose tangent-line slopes are determined by *g(x)*. For instance, the line tangent to *G(x)* at $x = 0.1$ should have the slope $g(0.1)$, or 0.2; the line tangent to *G(x)* at $x = 0.3$ should have the slope $g(0.3)$, or 0.6; and so on.

While we don't know yet what the graph of the function *G(x)* looks like, we can already draw a few of its tangent lines:

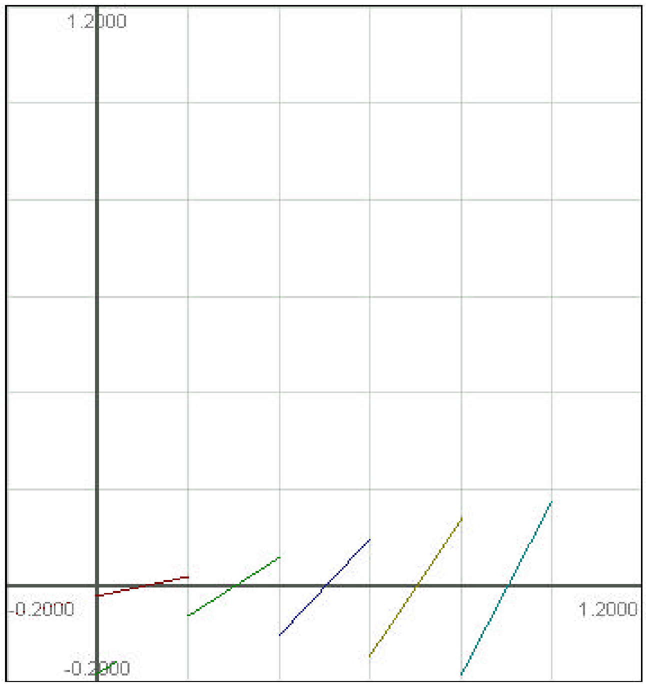

Figure 3. The Infancy of *G(x)*

Remember, we are not yet drawing *G(x)*. All we are plotting are some of its tangent lines, albeit vertically displaced. From the above graph you can see how the segment of *G(x)* that passes through $x = 0.1$ has a slope of $g(0.1)$, or 0.2; the segment that passes through $x = 0.3$ will have a slope of 0.6; and so on. We're on the right track!

However, the segments above clearly do not compose the function *G(x)* we are trying so hard to construct. **First**, the segments do not compose a continuous function. **Second**, most of the graph provides us with inaccurate information. For example, from looking at the above graph, you would think that the slope of line tangent to *G(x)* at $x = 0.35$ is 0.6; however, the correct slope should really be $g(0.35)$, or 0.7.

To solve both problems, we will need to 1) join the segments together and 2) make the segments significantly smaller. Let's solve the first problem first. Easily enough we have, in the graph below, connected the segments:

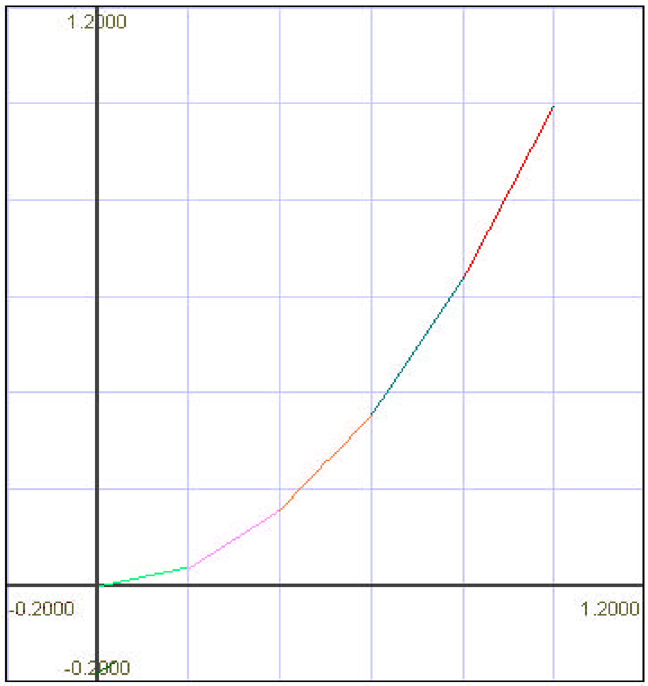

Figure 4. The Adolescence of *G(x)*

Before we move on to solving the second problem, let us calculate the function for the graph we have above. Let's say we wanted to determine the *y-value* for a given *x* on the above graph, what would we do? For example, imagine we wanted to know what the *y-value* was at $x = 0.6$. The easiest way to do this is probably to add up the heights of the three segments that lie between 0 and 0.6. To do that, we need to figure out the heights of each segment.

We all know from elementary school that $slope = \frac{rise}{run}$. Since we know the *run* of each segment, in our case it is 0.2, and we also know the *slope*, given by *g(x)*, we can easily solve for the *run*, or height, of each segment. In our case, the *y-value* at $x = 0.6$ would be $h_1 + h_2 + h_3$, where h_x is the height of segment x. Each h_x in turn is $\Delta x * m$, where Δx is the segment length, or the run, and m is the slope. That would translate into 0.2*0.2 + 0.2*0.6 + 0.2*1.0, or a final answer of 0.36. Looking at the graph on the previous page, the reader will see that the *y-value* at $x = 0.6$ is indeed 0.36. If we wanted to rewrite the above calculations in a more mathematical notation we could write:

$$(1) \quad \sum_{k=1}^{n} g(x_k)\Delta x$$

where n is the number of segments, Δx is the segment length (in our case 0.2), and x_k is the *x-value* for each segment k (in our example the x_k values were 0.1, 0.3, and 0.5).

Now, to complete the demonstration, we will attend to the second problem raised earlier. We noted that the segments were just too large; the slopes of the lines tangent to *G(x)* at points other than the segment midpoints did not equal *g(x)*. The *y-values* at points like 0.37 and 0.81 were incorrectly given as 0.6 and 1.8, when they should in fact have been 0.74 and 1.62. Thankfully, since we already have (1), resolving this problem is trivial. All we need to do is make the segment length Δx very, very small. That way, we'll be able to obtain the correct *y-value* for any given x. To do this, we'll just rewrite the above equation adding a limit that tells the reader that we want Δx to be as close to zero as possible:

$$(2) \quad \lim_{\Delta x \to 0} \sum_{k=1}^{n} g(x_k)\Delta x$$

Now the equation is a perfect tool to draw any function *G(x)* whose derivative is the given *g(x)*. We managed to do derivation "backwards!"

Graphing our latest version of the function, we get

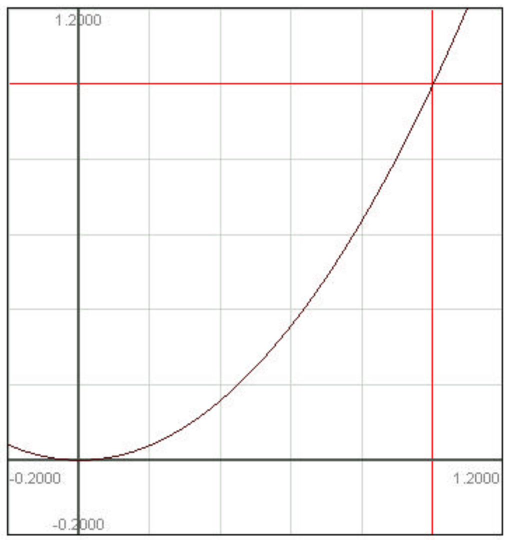

Figure 5. A Fully Developed *G(x)*

At this point I hope that the above function rings a bell. Aside from being the function that describes derivation "backwards," it is also the one that describes integration "forwards." In fact, it is the same equation that you'll reach when using Riemann's Sums to find the area under a curve!

Thus, in a brief few pages we have shown that the method for finding slopes of tangent lines is the "opposite" of the method for finding areas under a curve--and so we have the Fundamental Theorem of Calculus!