TEACHING NOTE 97-14:

BINOMIAL PRICING OF INTEREST RATE DERIVATIVES

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This teaching note shows how a binomial term structure can be used to price derivatives based on interest rates. It does not get into the issue of how to fit a binomial model of the term structure. It assumes that an arbitrage-free binomial model has already been derived. It further assumes that the derivatives are based only on the one-period interest rate. Accordingly, let us use the four period binomial model that is represented by the following tree containing the evolution of one-period rates. We shall use .5 as the up-state probability and .5 as the down-state probability. These are, of course, the risk neutral/equivalent martingale probabilities, not the actual probabilities.

Time 0	Time 1	Time 2	Time 3	Time 4
				r(4,5) = 16.72 %
			r(3,4) = 15.15 %	
		r(2,3) = 13.61 %		r(4,5) = 13.32 %
	r(1,2) = 12.06 %		r(3,4) = 11.80 %	
r(0,1) = 10.50 %		r(2,3) = 10.30 %		r(4,5) = 10.02 %
	r(1,2) = 8.80 %		r(3,4) = 8.54 %	
		r(2,3) = 7.09 %		r(4,5) = 6.82 %
			r(3,4) = 5.38 %	
				r(4,5) = 3.71 %

Recall that such a tree is developed by starting with the original term structure of zero coupon bond prices for of all maturities. In this case, those prices for $1 face value bonds of maturities of 1, 2, 3, 4 and 5 periods are $P(0,1) = 0.905$, $P(0,2) = 0.820$, $P(0,3) = 0.743$, $P(0,4) = 0.676$ and $P(0,5) = 0.615$. Thus, for example, the rate on a 1-period bond is $1/P(0,1) = 1/0.905 = 1.105$. In some applications, one converts this to a continuously compounded rate. In this case, however, we simply wish to start with the zero-coupon bond prices and one-period rates. The no-arbitrage argument is

used to fit the evolution of the prices of each zero coupon bond until its maturity. This is equivalent to the local expectations hypothesis, which is that the expected return using risk neutral probabilities on any bond over one time period is the same for all bonds.

The prices of coupon bonds are naturally obtained by adding up the prices of component zero coupon bonds. The prices of options, forwards, and futures on coupon and zero coupon bonds are easily obtained using the information on the evolution of the zero coupon bond prices. In this note, our focus is not on these derivatives on bonds, but rather derivatives on the one-period interest rate.

Forward Rate Agreements

A forward rate agreement or FRA is a forward contract in which at expiration one party makes a fixed interest payment and the other makes a payment determined by the rate at that time. Consider an FRA expiring at time 2. Obviously it could pay off 13.61 %, 10.30 %, or 7.09 %. What would be an appropriate fixed rate to agree on at the start?

First note that the payoff of an FRA is as follows:

$$\frac{Max(0, interest\ rate\ -\ fixed\ rate)}{1\ +\ interest\ rate}.$$

The expression "interest rate" is the underlying interest rate in the market at the time the FRA expires. As is the case with FRAs, the payment is made immediately, but is discounted by the current one-period interest rate. This reflects the fact that the interest rate in the market is a rate that will be paid one period later on the instrument to which it applies, in this case, a Eurodollar time deposit.

The fixed rate is set at time 0 such that the risk neutral expected payoff is zero. In this case, the fixed rate is the solution, F, to the following equation:

$$.5^2\left(\frac{0.1361\ -\ F}{1.1361}\right)\ +\ 2(.5)(.5)\left(\frac{0.1030\ -\ F}{1.1030}\right)\ +\ .5^2\left(\frac{0.0709\ -\ F}{1.0709}\right)\ =\ 0.0.$$

Note that $.5^2$ is the probability of the top state at time 2, $2(.5)(.5)$ is the probability of the middle state at time 2, and $.5^2$ is the probability of the bottom state at time 1. Solving gives F = .1028 or 10.28 %. Thus, if one agreed to enter into a two-period FRA, the contract would involve a commitment

to pay at time 2 a fixed rate of 10.28 % and receive the one-period floating rate that results at time 2, with the net differential discounted one period at the one-period floating rate..

The table below shows the value of the FRA during its life.

Time 0	Time 1	Time 2
		[0.1361 - 0.1028]/1.1361 = 0.0293
	[.0293(.5) + 0.0002(.5)]1.1206 = 0.0132	
[0.0132(.5) - 0.0136(.5)]/1.105 = -0.0002		[0.1030 - 0.1028]/1.1030 = 0.0002
	[0.0002(.5) - 0.0298(.5)]/1.0880 = 0.0136	
		[0.0709 - 0.1028]/1.0709 = -0.0298

As noted, at time 2, the payoff is the difference between the interest rate and the fixed rate on the FRA, discounted back one period at the current one-period rate. Stepping back to time 1, at each point the value of the FRA is the probability-weighted value of the next two possible values, discounted back by the current one-period rate. Thus, in the top state at time 2, the interest rate is 13.61% so the payoff is .1361 - .1028 and this amount is discounted back one period at the current one-period rate of 13.61%. The corresponding number in the middle outcome at time 2 is 0.0002. The top outcome at time 1 is a weighted average of 0.0293 and 0.0002, discounted back by the one-period rate at this point, which is 12.06%. Working back to the present, we obtain the current value of -0.0002. This is supposed to be zero, but differs slightly due to round-off errors.

For practice, determine the prices of FRAs of one, three and four periods. The answers are 10.41 %, 10.12 % and 9.97 %. The discounted values of these three FRAs at time 0 should be 0.0.

Interest Rate Caps and Floors

An interest rate cap is a series of independent call options on an interest rate while a floor is a series of independent put options on an interest rate. For example, a three-period cap with a strike rate of 10 % contains a call option on the rate with a strike rate of 10 % expiring at time 1, another call option on the rate with a strike rate of 10 % expiring at time 2, and a third call option on the rate with a strike rate of 10 % expiring at time 3. Thus, at each time point, if the rate is more than 10 %, there is an expiring call option paying off the rate minus 10 %. Exercise of any one call option does not affect one's right to exercise another. Common convention provides that when the option expires, the payoff is determined, but the actual payment is not made for one more period. This is in contrast to an FRA where expiration and payment are made at the same time. The effect of this factor is to require a discounting of the payment at expiration at the one-period rate. The deferral of the payment, while appearing to be a non-standard convention, actually is more appropriate because it corresponds to the manner in which floating interest payments on a loan are made. The rate is set at the beginning of the period and payment is made at the end.

The value of a cap or floor is the sum of the values of the component options, called caplets or floorlets. To value each caplet or floorlet we simply determine its value in each of the possible payoff states. We then step back and weight each of the next two outcomes by their probabilities and discounted this probability-weighted value back one step at the current one-period rate.

Consider a four-period cap on the one-period rate struck at 9 %. The cap consists of four component caplets, one expiring at time 1, one at time 2, one at time 3, and one at time 4. On each caplet expiration date, the cap pays off the one-period rate minus the strike rate if the former is higher and zero if the latter is higher.

Let us first value the four-period caplet. The formula for the payoff of a caplet is

$$\frac{Max(0, interest\ rate\ -\ exercise\ rate)}{1\ +\ interest\ rate}.$$

The discounting in the denominator is because the payoff is made one period later. In other words, if "interest rate - exercise rate" is positive, the payoff is made one period later. Hence, we discount by the current one-period rate.

In this case, the payoffs of the caplet are Max(0,0.1672 - 0.09)/1.1672 = .0661, Max(0,0.1332 - 0.09)/1.1332 = 0.0381, Max(0,0.1002 - 0.09)/1.1002 = 0.0093, Max(0,0.0682 - 0.09)/1.0682 = 0.0, and Max(0,0.0371 - 0.09)/1.0371 = 0.0. To value the caplet at each node at time 3, we simply discount the expected payment based on the next two possible nodes. Thus, in the highest state at time 3, the four-period caplet value is [0.0661(.5) + 0.0381(.5)]/1.1515 = 0.0452. In the second highest state at time 3, the value is [0.0381(.5) + 0.0093(.5)]/1.1180 = 0.0212. The values at the lower two states are [.5(0.0093) + .5(0.0)]/1.0854 = 0.0043 and 0.0. Proceeding in this manner, we would obtain the value at time 0 of 0.0111, as shown below:

Time 0	Time 1	Time 2	Time 3	Time 4
				Max(0,0.1672 - 0.09)/1.1672 = 0.0661
			[0.0661(.5) + 0.0381(.5)]/1.1515 = 0.0452	
		[0.0452(.5) + 0.0212(.5)]/1.1361 = 0.0292		Max(0,0.1332 - 0.09)/1.1332 = 0.0381
	[0.0292(.5) + 0.0116(.5)]/1.1206 = 0.0182		[0.0381(.5) + 0.0093(.5)]/1.1180 = 0.0212	
[0.0182(.5) + 0.0063(.5)]1.1050 = 0.0111		[0.0212(.5) + 0.0043(.5)]/1.1030 = 0.0116		Max(0,0.1002 - 0.09)/1.1002 = 0.0093
	[0.0116(.5) + 0.0020(.5)]/1.0880 = 0.0063		[0.0093(.5) + 0.0(.5)]/1.0854 = 0.0043	
		[0.0043(.5) + 0.0(.5)]/1.0709 = 0.0020		Max(0,0.0682 - 0.09)/1.0682 = 0.0000

			$[0.0(.5) +$ $0.0(.5)]/1.0538$ $= 0.0$	
				Max(0,0.0371 - 0.09)/1.0371 = 0.0000

This procedure obtained only the value of the four-period caplet. A four-period cap consists of that option plus options expiring at times 3, 2 and 1. The caplet expiring at time 3 is worth 0.0116. The caplet expiring at time 2 is worth 0.0130 and the caplet expiring at time 1 is worth 0.0124. Thus, the four-period cap is worth 0.0111 + 0.0116 + 0.0130 + 0.0124 = 0.0481. From these results, we can see that a three-period cap would be worth 0.0116 + 0.0130 + 0.0124 = 0.0370, a two-period cap would be worth 0.0130 + 0.0124 = 0.0254, and a one-period cap would be worth 0.0124.

A floor would be valued the same way except that the payoffs at the expiration of each floorlet would be structured as a put,

$$\frac{Max(0, exercise\ rate\ -\ interest\ rate)}{1\ +\ interest\ rate}.$$

The values of the four floorlets expiring at times 1, 2, 3 and 4 would be 0.0050, 0.0057, 0.0081 and 0.0080.

Most caps and floors are designed to hedge specific exposure to interest rate adjustments on floating rate loans. Thus, they tend to be European-style options. American caps and floors can be easily priced, however, by replacing any state value with its exercise value if the latter is higher. Keep in mind that if an early exercise decision is made, the payment is still delayed one period. Otherwise, interest rate options would always be exercised whenever they are in-the-money.

Interest Rate Swaps

The plain vanilla interest rate swap is a series of fixed payments in exchange for a series of floating payments based on the one-period rate. When the floating rate is determined, however, the

payment is delayed one period, as in the case for options. Pricing a swap, meaning to determine the fixed payment, does not require that we model the full evolution of the term structure. Rather it requires only the initial term structure of zero coupon bonds. The fixed payment on the swap equates the present value of the fixed payments to the present value of the floating payments. This is the same as solving for the fixed rate on a par value bond. Thus, for a $1 three-period par swap, we must solve the equation,

$$F[0.905 + 0.820 + 0.743] + 1.0[0.743] = 1.0.$$

The terms in brackets are the present value factors, or actually the zero coupon bond prices, for one, two and three periods. The left-hand side is the present value of a fixed rate par bond and the right-hand side is the present value of a floating rate bond. In a swap the principal payments are not exchanged, but adding them on both sides of the pricing equation simplifies things. The solution is

$$\frac{1 - 0.743}{0.905 + 0.820 + 0.743} = 0.1041.$$

Remember that the value of a swap is zero at the start, but the value changes as interest rates change. Thus, we can use the evolution of the term structure to value the swap at various points in its life. If everything is consistent, we should obtain a value of zero at the present. The table below shows the payments and the value of the swap. Note, however, that these are not positioned when the payments are made, but rather when the payments are determined. Each payment is deferred one period. In other words, the payment is determined at the beginning of a period and payment is made at the end of a period. Thus, for a three-period swap, we need to show the binomial tree only up to time 2.

The value of 0.0306 in the top state at time 2 is really worth $0.0320/1.1361 = 0.0282$. The remaining values at time 2 are shown in the parentheses. In the top state at time 1, the payment value of $0.1206 - 0.1041 = 0.0165$ is added to the expected value one period later and discounted to obtain $[0.0165 + 0.0282(.5) - 0.0010(.5)]/1.1206 = 0.0269$. The remaining computations are shown in the cells in the table. At each point we simply add the value of the payment determined at that point to the expected value of the two possible upcoming swap values. We then discount that total, reflecting

the fact that the current payment is not received until one period later. The overall value of the swap at time 0 should be 0.0, the discrepancy here arising only from a rounding error.

Time 0	Time 1	Time 2
		[0.1361 - 0.1041]/1.1361 = 0.0282
	[(0.1206 - 0.1041) + 0.0282(.5) - 0.0010(.5)]/1.1206 = 0.0269	
[(0.1050 - 0.1041) + .0269(.5) - 0.0295(.5)]/1.105 = -0.0004		[0.1030 - 0.1041]/1.1030 = -0.0010
	[(0.0880 - 0.1041) - 0.0010(.5) -0.0310(.5)]/1.0880 = -0.0295	
		[0.0709 - 0.1041]/1.0709 = -0.0310

Interest Rate Swaptions

A swaption is an option to enter into a swap. There are two types of swaptions. A payer swaption is an option to enter into a swap as a fixed rate payer, which implies floating rate receiver. A receiver swaption is an option to enter into a swap as a fixed rate receiver, implying that you would pay floating. A swaption is always based on an underlying swap and has a strike rate, which determines the fixed rate that can be obtained when the swaption is exercised and the underlying swap is initiated.

A payer swaption, allowing the holder to enter into a swap as a fixed rate payer, will be most appropriate when higher interest rates are expected. If higher interest rates do occur, the rate on the underlying swap will increase, giving the payer swaption the potential for expiring in-the-money. Consequently, payer swaptions are somewhat like interest rate call options, and can be viewed as variations of put options on bonds. In fact there is a direct relationship between the price of a put option on a bond and a payer swaption. Similarly a receiver swaption is most appropriate when rates are expected to fall and is, thus, somewhat like an interest rate put option or a call option on a bond.

Let us price a two-period European-style payer swaption with a strike rate of 10.50 % where the underlying swap is a three-period swap. This swaption will allow us to enter into a three-period swap at time 2, paying a fixed rate of 10.50 %. In order to determine the three possible payoffs at

time 2, we must solve for the fixed rate on a three-period swap at time 2 in each node. To obtain this information we must know the prices of one-, two- and three-period zero coupon bonds in each state at time 2. We have not previously seen this information, but it would be obtained when we fit the binomial term structure. The prices of one-, two-, and three-period zero coupon bonds at time 2 are, respectively, 0.881, 0.778, and 0.687 in the upper state, 0.908, 0.825, and 0.751 in the middle state, and 0.935, 0.875 and 0.821 in the lower state. Following the same procedure for pricing the swap as previously described, we would obtain the fixed rate on a three-period swap in the three states at time 2 as

$$\frac{1 - 0.687}{0.881 + 0.778 + 0.687} = 0.1334$$

$$\frac{1 - 0.751}{0.908 + 0.825 + 0.751} = 0.1002$$

$$\frac{1 - 0.821}{0.935 + 0.875 + 0.821} = 0.0680.$$

Consider the upper state at time 2. The swaption expires and the fixed rate on the underlying swap is 0.1334. What is the value at this point of the swaption? Exercising the swaption allows us to enter into a 3-period swap paying a fixed rate of 0.1050. In the market, the swap could be initiated at a fixed rate of 0.1334. Thus, we could exercise the swaption, entering into a swap to pay fixed of 0.1050 and enter into a swap in the market to receive fixed of 0.1334. The floating sides of the two swaps cancel. We have, therefore, created a three-period annuity of 0.1334 - 0.1050 = 0.0284, starting at time 4. The value at time 3 of this annuity is, therefore, 0.0284(0.881 + 0.778 + 0.687) = 0.0666, where the terms in parentheses are the one-, two- and three-period bond prices, which are the discount factors. In the middle state, the value of the swaption is zero, because the strike rate is 0.1050 and the market rate is 0.1002. Obviously, in the lower state, the value is also zero. So in general the formula is

$$Max(0, \text{Swap rate} - \text{exercise rate}) \sum \text{PV factors.}$$

Now we simply proceed backwards through the tree, determining the discounted probability-weighted value of the option at each state. In the upper state of time 1, we have [0.0666(.5) +

0.0(.5)]/1.1206 = 0.0297. In the lower state of time 1, we have an obvious value of 0.0, since the swaption would be worth 0.0 in each of the two states at time 2 that we could get to from this state. The value of the swaption at time 0 is, thus, [0.0297(.5) + 0.0(.5)]/1.1050 = 0.0134.

Valuation of the swaption is shown in the table below.

Time 0	Time 1	Time 2
		Max(0,0.1334 - 0.1050)(0.881 + 0.778 + 0.687) = 0.0666
	[0.0666(.5) + 0.0(.5)]/1.1206 = 0.0297	
[0.0297(.5) + 0.0(.5)]1.105 = 0.0134		Max(0,0.1002 - 0.1050)(0.908 + 0.825 + 0.751) = 0.0
	[0.0(.5) + 0.0(.5)]/1.0880 = 0.0297	
		Max(0,0.0680 - 0.1050)(0.935 + 0.875 + 0.821) = 0.0

If the swaption were American-style, it would be subject to early exercise in each state. In fact we would find that in the time 1 upper state, we would exercise it. To arrive at this result, we need to know the discount factors for one-, two-, and three-period bonds in that state, which are 0.893, 0.798, and 0.715. In that state, the three-period swap rate would be 0.1185. The value of the swaption, if exercised early at that point, would be (0.1185 - 0.1050)(0.893 + 0.798 + 0.715) = 0.0325. This exceeds the European-style value of 0.0297. Thus, we would replace 0.0297 with 0.0325. The value of the swaption today would, thus, be (0.0325(.5) + 0.0(.5))/1.1050 = 0.0147.

Receiver swaptions would be priced in the same manner except that the payoffs at expiration would be based on the establishment of a swap paying the fixed rate. Their payoffs are determined by the formula

$$Max(0, \text{Exercise rate} - \text{Swap rate}) \sum \text{PV factors.}$$

References

The binomial model is widely covered in derivatives books, but not all of them cover binomial term structure modeling of interest rate derivatives. The following books do give such coverage:

Arditti, Fred D. *Derivatives: A Comprehensive Resources for Options, Futures, Interst Rate Swaps, and Mortgage Securities*. Boston: Harvard Business School Press (1996), Ch. 7.

Chance, Don M. *An Introduction to Derivatives and Risk Management*. Fort Worth: Harcourt, Inc. (2001), Ch. 14.

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Tuckman, Bruce. *Fixed Income Securities: Tools for Today's Markets*. New York: John Wiley (1996), Chs. 5-8.