Global Financial Management

Portfolio Analysis and Diversification

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4.0 Overview

This class provides an overview of individual asset allocation. It is shown that an individual can reduce the risk of his portfolio without sacrificing any expected return simply by spreading his wealth over a number of assets in an appropriate way. This technique of diversification is explained in some detail in terms of a simple two-asset example in order to build intuition. This section provides the main tools for understanding asset pricing models.

4.1 Objectives

After completing this class, you should be able to:

- Explain the concept of risk aversion
- Distinguish between risk averse, risk neutral and risk loving investors and describe their behavior
- Compute the expected return of a portfolio
- Compute the variance and standard deviation of the return of a portfolio
- Compute the covariance and correlation of the returns of two assets
- Find the composition of the minimum-variance two-asset portfolio
- Explain the concept of diversification
- Explain how to construct a diversified portfolio in practice
- Explain the concept of an efficient frontier and the mean-variance frontier
- Analyze portfolios with a risk free asset

4.2 Introduction

So far in the course, we have not established any benchmarks to compare securities or portfolios other than expected returns. It is impossible to judge the quality of an investment by simply looking at its expected returns. For example, consider an advertisement from *The Wall Street Journal* which followed the following performance comparison.

The Franklin Income Fund	Dow Jones Industrial Average	Salomon's High Grade Bond Index	Cost of Living
516%	384%	283%	169%

These average returns over the past 15 years are higher than the *Dow Jones Industrial Average* and *Salomon's* High Grade Bonds. Does this mean that we can beat the market by investing in Franklin now? The answer is no. The advertisement tells us nothing about the risk of the *Franklin Fund*. We will always want to consider risk as well as return. *The Franklin Fund* stocks may be very risky and the only way people will hold the component stocks is to have a high expected return. So we have go beyond returns and develop a model of risk that allows us to compare stocks and portfolios.

4.3 The Planning Problem

We analyze portfolio allocation from the point of view of an individual consumer. The objective of this consumer is to maximize expected utility of consumption today, plus the expected utility of wealth, E[U(W)], (future consumption) tomorrow. Hence, the consumer has to make two decisions:

- How much should he consume today, and how much of his wealth should he invest in order to be able to consume more tomorrow.

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- What is the optimal way to invest that portion of his wealth that is not consumed today.

We will focus on the second part of the consumer problem here. We will also asume that all consumers are tional utility maximizers with a concave utility function. For simplicity, we write the consumer's utility as a simple function of his wealth, U(W).

The consumer's choice variables are: firstly, his consumption today, c_0, and secondly, his investment proportions in N assets, w_i. We will assume that one of these assets, $i = 0$, is risk free. So the consumer will use his initial wealth for either consumption today or investment in assets that pay off in the future. The consumer is constrained by his budget constraint that prescribes that he cannot invest and consume more than his total wealth:

$$W_0 = c_0 + \sum_{i=1}^{N} w_i p_i \tag{1}$$

This just says that the sum of today's consumption plus investment in the N assets with price P_i, cannot exceed initial wealth. Tomorrow's wealth will be determined by the payoffs on the investment strategy. Because these payoffs or returns are random (except for the risk free asset), tomorrow's wealth and the consumer's utility will also be random. We can therefore not say that the consumer maximizes his future utility, since this utility is not known. Rather we assume that consumers maximize *expected* utility. Hence, the consumers planning problem can be formulated as follows:

$$Max_{w_i} E(U(W))$$
$$s.t. \text{ condition (1)}$$

(2)

This is mathematical notation and means in words: the consumer tries to choose his portfolio allocation (the proportions w_i of his wealth he puts in asset i) so as to maximize his expected utility from consuming his wealth in the future. In order to find a solution to this problem we need to understand some properties of utility functions and the concept of risk aversion.

4.4 Risk Aversion

We characterize the utility function U(W) investors try to maximize by two key assumptions:

- Investors prefer more to less: their utility increases with wealth
- Investors utility increases with wealth at a decreasing rate

In mathematical notation these assumptions can be written as:

$$\frac{\partial U}{\partial W} > 0 \quad \frac{\partial^2 U}{\partial W^2} < 0$$

(3)

These are conditions on the form of the utility function. The first derivative being greater than zero means that you prefer more wealth to less wealth. This is a property that is not controversial. The second derivative being negative means that you prefer more to less at a decreasing rate as wealth gets larger. So you get more utility from a $10,000 increase in your wealth if your previous wealth was $20,000 rather than if your wealth was $2,000,000.

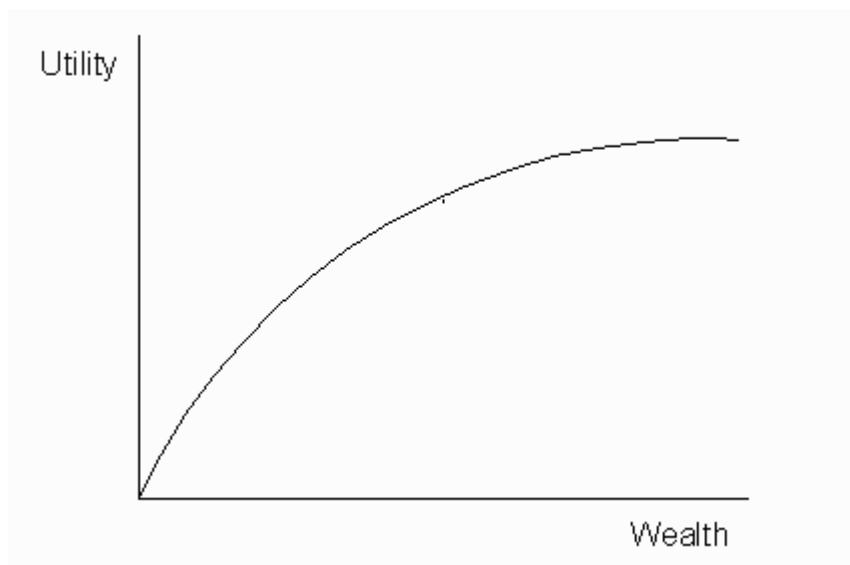A standard utility function that exhibits these two traits is illustrated in *Figure 1*.

Figure 1

The fact that the utility function is upward sloping indicates that the investor prefers more to less, no matter how wealthy he might become. The fact that the utility function is concave is an implication of the fact that the slope of the utility function is decreasing. The second property is important in order to understand risk aversion.

In order to illustrate the concept of risk aversion, consider an investor who is faced with a gamble whereby he bets $100 on the toss of a coin. If it's heads he wins $200, if it's tails he gets nothing. This gamble creates a 50/50 chance of increasing or decreasing his wealth by $100, so his expected wealth is unchanged. Risk aversion implies that the investor will actually reject such a fair bet.

Suppose the wealth of the investor if he rejects the bet is $100. For concreteness, suppose that the utility function of the investor has a simple functional form and $U(W) = \sqrt{W}$. Then we have:

	Accept bet		Reject bet W=100		Reject bet W=50	
	W	$\sqrt{W}$	**W**	$\sqrt{W}$	**W**	$\sqrt{W}$
Outcome 1	200	14.14	100	10	50	7.07
Outcome 2	0	0	100	10	50	7.07
Expected	100	7.07	100	10	50	7.07

In the example, we see that relative to the utility of initial wealth $U(W_0)=\sqrt{100}=10$, winning 100, provides a utility gain of 4.14, but losing 100 generates a utility loss of 7.07. Suppose the investor has to take the bet. Then we can see that he would pay an insurance premium up to 50 to have this risk removed: he is indifferent between a bet that pays off 200 with probability 0.5 and having 50 for sure. In other words the investor strictly prefers a safe payoff of 100 to a risky payoff with an expectation of 100, and he is indifferent between the risky bet with expected payoff of 100 and a safe payoff of 50. Hence, if his initial wealth is 100 and he had to take the bet, he would be willing to pay an insurance premium of 50 to have the risk of the bet removed.

Example 1

What if the gamble is not fair, so that the odds are stacked in the investor's favor? Whether an individual takes the gamble or not depends on his utility function, and on his initial wealth. First, suppose the individual has utility, $U(W)=W^{0.5}$, initial wealth of 20, and is faced with the following investment proposal:

Outcome	Probability
+15	50%
-10	50%

In this case, the investment proposal is better than a fair bet. What is the expected utility after undertaking the investment? We denote by p_u the probability that the investors wealth moves up, and by $p_d=1-p_u$ the probability that the investor's wealth moves down.

$$E[U(W)] = p_u U(W_0 + 15) + p_d(W_0 - 10)$$
$$= 0.5U(20 + 15) + 0.5U(20 - 10)$$
$$= 0.5\sqrt{35} + 0.5\sqrt{10}$$
$$= 0.5(5.916) + 0.5(3.16)$$
$$= 4.54$$

His current utility (without the investment) is:

$$U(W_0) = U(20) = (20)^{0.5} = 4.47$$

This example illustrates the expected utility rule: If the expected utility $E[U(W)]$ from owning a proposed investment exceeds the current utility $U(W_0)$ without the investment, then the investment should be undertaken.
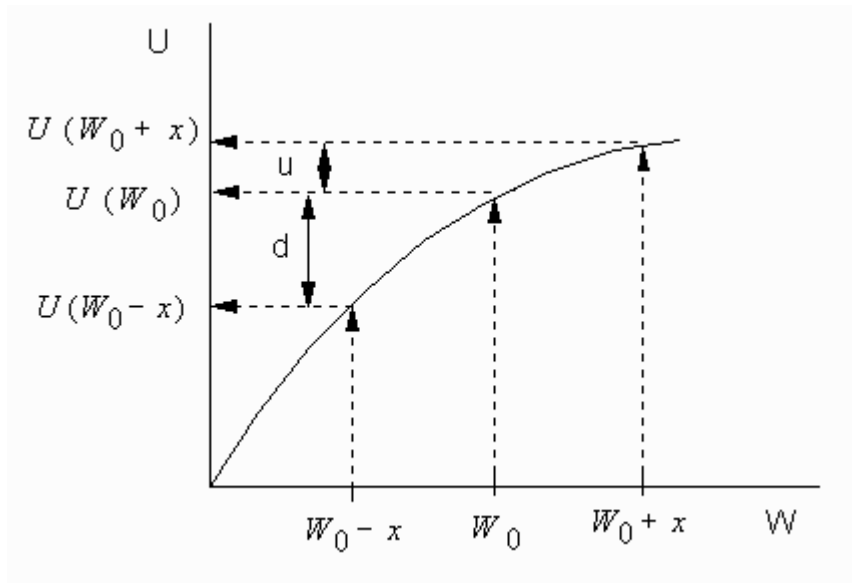

Figure 2

In example 1, since $E[U(W)]=4.54>U(W_0)=4.47$, the investor should accept the investment proposal.

Consider figure 2. The investor would be pleased if the outcome turns out to be *x*, which is illustrated by the increase in utility on the graph *U*. However, the investor would be extremely displeased if the outcome turns out to be -*x*. The decrease in his utility *d* when the outcome is -*x* is far greater than the increase in his utility *u* when the outcome is +*x*. This discussion illustrates a general principle:

A risk averse individual will never take a fair bet.

Example 2

Consider another example where the investor has initial wealth of 20.5 and considers investing in an asset that increases his wealth by 4.5 with probability p_u, and decreases his wealth by 4.5 with probability $p_d=1-p_u$.
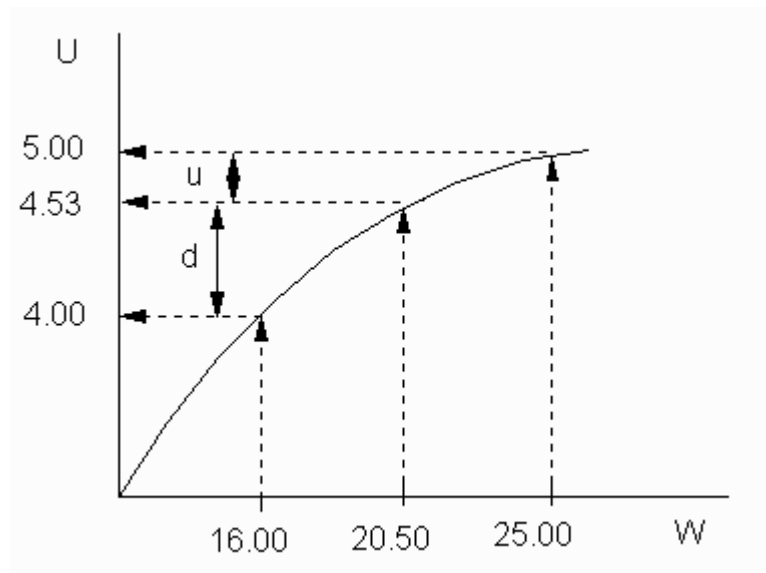

Figure 3

Then his utility from buying the investment is:

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$$E[U(W)] = p_u U(20.5 + 4.5) + (1 - p_u)U(20.5 - 4.5)$$
$$= p_u\sqrt{25} + (1 - p_u)\sqrt{16}$$
$$= 4 + p_u$$

Conversely, rejecting the investment gives:

$$U(20.5) = \sqrt{20.5} = 4.53$$

Hence, the investment is optimal if and only if the probability of the increase is at least $p_u=53\%$. Hence, the investor would reject a fair bet ($p_u=0.5$), accept any bet with $p_u>53\%$, and is indifferent if $p_u=53\%$.

Example 3

Reconsider example 1 and suppose that the individual's utility function is logarithmic so that U(W)=ln W. What is the expected utility after undertaking the investment?

$$E[U(W)] = p_u U(W_0 + 15) + p_d U(W_0 - 10)$$
$$= 0.5U\ln(35) + 0.5\ln(10)$$
$$= 0.5(3.555) + 0.5(2.30)$$
$$= 2.93$$

His current utility (without the investment) is:

$$U(W_0) = U(20) = \ln(20) = 3.00$$

Therefore, since E[U(W)]<U(W), the investor should not accept the investment proposal.

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Finally, suppose that the investor still has log utility, but that his initial wealth is 100(rather than 20). In this case, the expected utility after undertaking the investment is:

$$E[U(W)] = p_u U(100+15) + pdU(100-10)$$
$$= 0.5\ln(115) + 0.5\ln(90)$$
$$= 0.5(4.745) + 0.5(4.5)$$
$$= 4.622$$

The investor's current utility (without the investment) is $U(W_0)=\ln(100)=4.60$. Therefore, since $E[U(W)]>U(W)$, the investor should accept the investment proposal if his initial wealth is 100.

This example indicates that different investors have different attitudes toward risk, which is captured in the form of differently shaped utility functions. Moreover, a single individual, whose utility function does not change can become more tolerant toward risk as his wealth increases.

4.5 Certainty Equivalent

The certainty equivalent C is the minimum amount of cash that an investor would accept in exchange for all of his investments and his initial wealth. Having $\$C$ cash will make the investor as well off as having his initial wealth and his investment portfolio. Therefore:

$$E[U(W)] = U(C) \tag{4}$$

Example 4

Suppose an investor, who has square root utility and initial wealth of $100,000, is deciding whether or not to buy car insurance. The outcomes and associated probabilities, based on the investor's driving record are:

Event	Loss	Probability
No Accident	0	p_u=95%
Accident	-50,000	p_d=5%

First, determine the investor's expected utility if he does not buy insurance:

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$$E[U(W)] = p_u U(W_0 + 0) + p_d U(W_0 - 50{,}000)$$
$$= 0.95\sqrt{100{,}000} + 0.05\sqrt{50{,}000}$$
$$= 0.95(316.227) + 0.05(223.607)$$
$$= 300.415 + 11.18$$
$$= 311.596$$

Next, compute the certainty equivalent so that the investor is just as happy with $\$C$ cash as with having $100,000 and taking the risk of the accident. That is, $\$C$ solves:

.

$$E[U(W)] = U(C) = \sqrt{C}$$
$$311.60 = \sqrt{C}$$
$$311.60^2 = C$$
$$C = 97{,}092.07$$

The investor originally had $100,000 ($W_0$) plus the risk of an accident. If he had $97,092.07 and was indemnified against the cost of an accident, he would be just as happy. Therefore, he is prepared to pay $100,000-$97,092.07=$2,907.93 for car insurance.

4.6 Attitudes towards Risk

There are three prevailing attitudes toward risk; risk averse, risk loving, and risk neutral.

- The risk averse investor possesses a certainty equivalent less than the expected payoff from a given bet. She rejects all fair bets and is willing to pay a premium for insurance.

- The risk loving investor possesses a certainty equivalent greater than their expected payoff from a given bet. She accepts all fair bets and never purchases insurance.

- The risk neutral investor possesses a certainty equivalent equal to the expected payoff from a given bet. She is indifferent about fair bets and does not pay a premium for insurance.
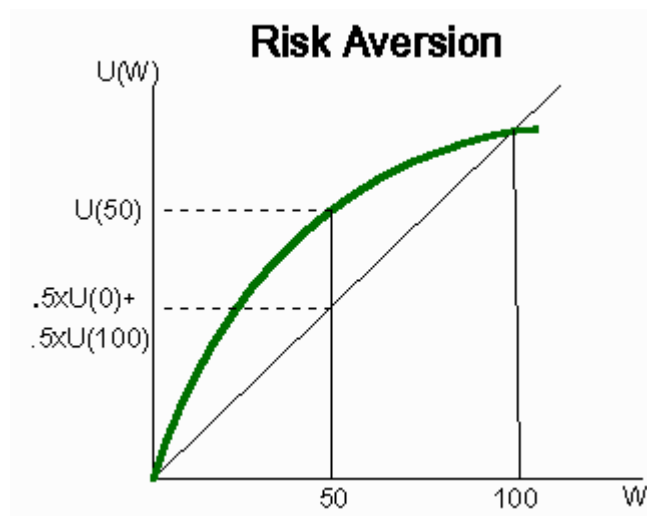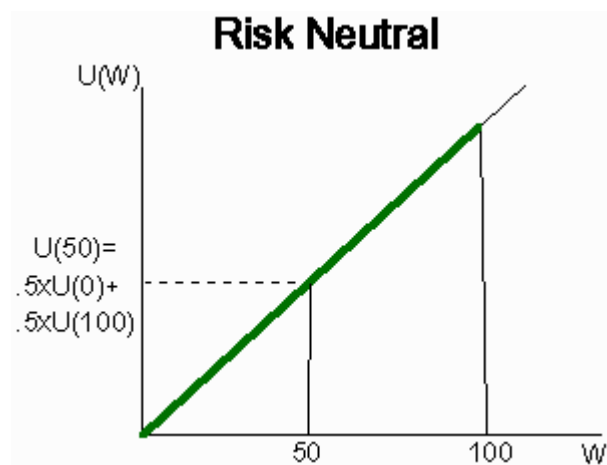
Figure 4

Figure 5
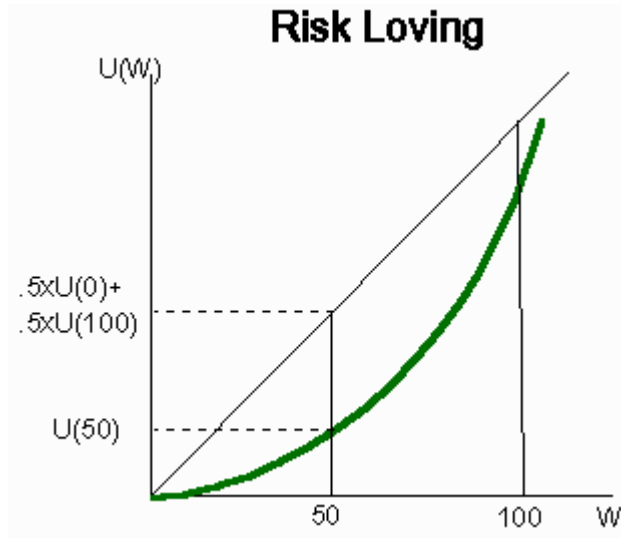
Figure 6

4.7 Portfolio Selection: The Two-Asset Case

Portfolios of assets can be compared on the basis of their risk and return characteristics. In this section we consider tools for analyzing portfolios. We need to know how to measure the risk and return characteristics of a given portfolio of securities.

The expected return of a two asset portfolio is simply a weighted average of the expected returns of each asset in the portfolio.

$$E[r_p] = w_1 E[r_1] + (1 - w_1) E[r_2] \tag{5}$$

where w_1 and $(1-w_1) = w_2$ are the percentage of portfolio value invested in each asset.

The variance of the portfolio can be written as:

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$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$$

$$or \tag{6}$$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2}$$

Recall the formula for correlation. The correlation of portfolios with returns *a* and *b* is just the covariance divided by the product of the standard deviations.

$$\rho_{i,j} = \frac{COV(r_i, r_j)}{\sqrt{VAR(r_i)VAR(r_j)}} = \frac{COV(r_i, r_j)}{\sigma_i\sigma_j} \tag{7}$$

Hence, we have $Cov(r_1, r_2) = \sigma_{1,2} = \rho_{12}\sigma_1\sigma_2$. We see immediately that the riskiness of the portfolio depends on (1) the riskines of each individual asset in the portfolio, (2) on the weights put into each of the risky assets, and (3) on the correlation between the assets. Consider the following three portfolios, all of which are equally weighted between two risky assets that have exactly the same standard deviation.

Example 5- Perfect Positive Correlation $(\rho = 1)$

The first case of interest is that of perfect positive correlation. Using the formula:

$$\sigma_p^2 = 0.25\sigma^2 + 0.25\sigma^2 + (2 \cdot 05 \cdot 0.5 \cdot \sigma^2 \cdot 1) = \sigma^2 \tag{8}$$

This result demonstrates that the portfolio variance is the same as the variance for each asset. So diversification does not reduce the portfolio variance relative to a portfolio that is completely invested in one asset in this case.

Example 6 – No Correlation $(\rho = 0)$

The second case of interest is that of zero correlation. Again, plugging into the formula:

$$\sigma_p^2 = 0.25\sigma^2 + 0.25\sigma^2 + \left(2 \cdot 0.5 \cdot 0.5 \cdot \sigma^2 \cdot 0\right) = 0.5 \cdot \sigma^2 \qquad (9)$$

This result demonstrates that the portfolio variance is half of the variance of the individual assets. So combining stocks that have less than perfect positive correlation is a strategy that will reduce the variance of the returns on your portfolio. This is called *diversification.*

Example 7 – Perfect Negative Correlation $\left(\rho = -1\right)$

If two assets could be found which have perfect negative correlation, then we combine these assets to create a risk free portfolio:

$$\sigma_p^2 = 0.25\sigma^2 + 0.25\sigma^2 + \left(2 \cdot 05 \cdot 05 \cdot \sigma^2 \cdot (-1)\right) = 0.0 \qquad (10)$$

These two assets create a perfect hedge. This shows that diversification can be thought of as a hedge of risks.

The following graph tells the story. Suppose we randomly selected a stock and plotted its standard deviation. Now we randomly select another stock and plot the standard deviation of the equally weighted portfolio. We continue the exercise. Just by randomly selecting stocks that are not perfectly correlated we can decrease portfolio variance.
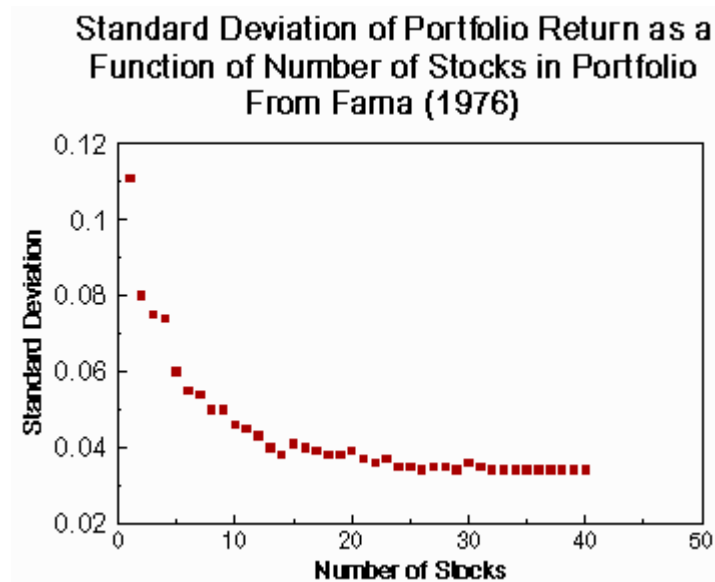
Figure 7

4.8 Efficient Portfolios

Our discussion of utility functions and risk aversion provided two conclusions. First, consumers prefer more to less. In terms of a security or portfolio, consumers prefer more return to less return. Second, consumers prefer less variance to more variance. Remember that the risk averse consumer will always turn down a fair bet. In terms of a security or portfolio, the consumer will prefer a portfolio with less variance to another higher variance portfolio with an equal expected return.

These insights lead to two rules of portfolio selection.

- Any investor who chooses to hold a portfolio with variance σ_p^2 will want the portfolio that has the maximum mean return possible among those portfolios that have variance σ_p^2.

- Similarly, any investor who chooses a portfolio with mean $E[r_p]$ will want the portfolio with the minimum variance possible among those with mean $E[r_p]$.

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A portfolio that satisfies these conditions is known as an **efficient** portfolio. A portfolio is inefficient if there exists another portfolio with:

a. $E[r_p] > E[r]$ and $\sigma_p^2 \leq \sigma^2$ or
b. $E[r_p] \geq E[r]$ and $\sigma_p^2 < \sigma^2$

We can summarize (a) and (b) in words:

No other portfolio with the same, or higher, expected return has a lower standard deviation of return.

No other portfolio with the same, or lower, standard deviation of return has a higher expected return.

There are typically many efficient portfolios, and the graph of all efficient portfolios is also referred to as the **efficient frontier.** Generally, different investors choose different portfolios from the efficient frontier, but all risk averse investors will choose some portfolio on the efficient frontier. We now proceed to characterize the efficient frontier for different correlations.

Case 1: Perfect Positive Correlation $(\rho = 1)$

Consider a number of simple cases. First, if the correlation between the two securities is one, then the standard deviation on the portfolio is:

$$\sigma_p = \left| w\sigma_1 + (1-w)\sigma_2 \right| \tag{11}$$

because

$$\begin{aligned}
\sigma_p^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 \\
&= \left(w_1\sigma_1 + w_2\sigma_2\right)^2 \\
&= \left(w\sigma_1 + (1-w)\sigma_2\right)^2
\end{aligned} \tag{12}$$

This result has a simple geometric interpretation. Setting the portfolio variance equal to zero and solving for w from (12) gives:

$$w = \frac{\sigma_P - \sigma_2}{\sigma_1 - \sigma_2} \tag{13}$$

Substituting into the equation for expected returns gives:

$$E[r_P] = E[r_2] + \frac{\sigma_1 - \sigma_P}{\sigma_1 - \sigma_2}\left(E[r_1] - E[r_2]\right) \tag{14}$$

Hence, in this special case, the portfolio return is a linear function of the portfolio standard deviation. We can plot the combinations of risk and return in mean-standard deviation space yielding portfolios that lie on a straight line.[1]

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[1] Note that we are plotting *standard deviations* on the x-axis while in our discussion we have used the *variance* of the portfolio return. This is just a matter of scaling. The graph would not change qualitatively if we were to plot the variance on the x-axis. Consequently, throughout the rest of the section we will continue to discuss the risk-return tradeoff using the variance of the portfolio return although our plots will depict the portfolio standard deviation.
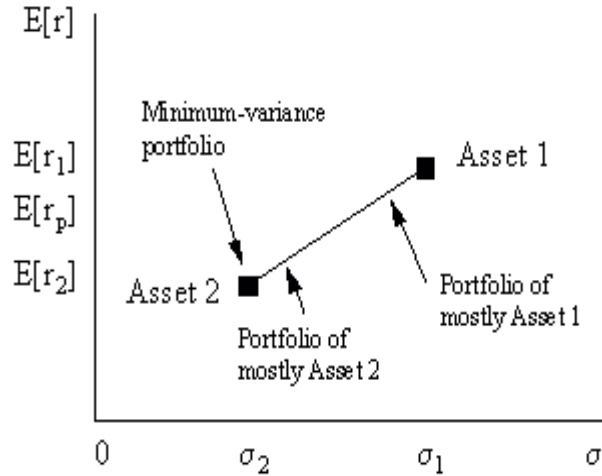
Figure 8

Note that diversification when the correlation between the securities is one is ineffective: we cannot increase the portfolio mean and reduce portfolio standard deviation at the same time.

Case 2 Perfect Negative Correlation $\left(\rho = -1\right)$

The second straightforward situation is when the correlation between the two securities is negative one. The standard deviation of the portfolio is:

$$\sigma_p = \left| w\sigma_1 - (1-w)\sigma_2 \right| \tag{15}$$

This immediately implies that we can drive the standard deviation of the portfolio to zero by choosing the right weights. Setting the left-hand side equal to zero, we can solve for w:

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2} \tag{16}$$

This point corresponds to the point *y* on the diagram. The expected return of this portfolio will be:

$$E[r_p] = \left(\frac{\sigma_2}{\sigma_1 + \sigma_2}\right)E[r_1] + \left(\frac{\sigma_1}{\sigma_1 + \sigma_2}\right)E[r_2] \qquad (17)$$
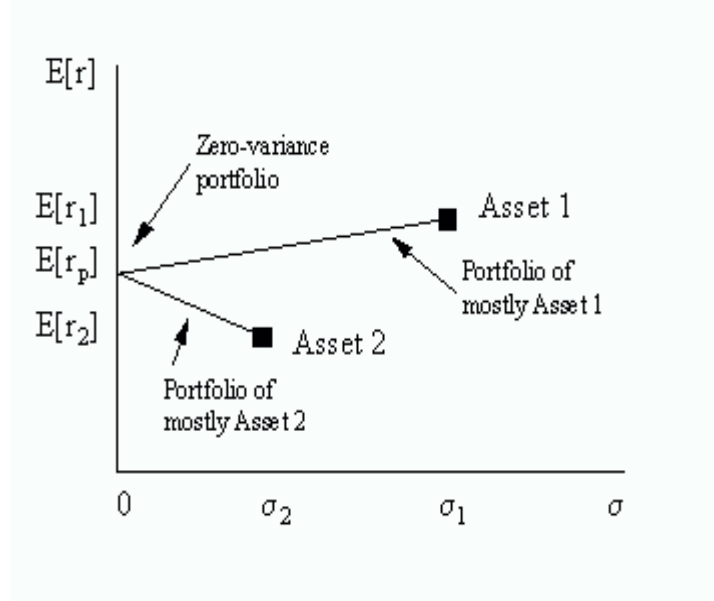
Figure 9

Case 3 Correlation between -1 and +1

Finally, consider the general case where the returns of the two assets are neither perfectly positively nor perfectly negatively correlated. Without loss of generality, assume that *Asset* 2 has lower variance than *Asset* 1. In this case, two results are possible. If there exists some portfolio of *Assets* 1 and 2 that has higher expected return and lower variance than *Asset* 2, then there are gains from diversification. The possible portfolios lie on a parabola that has a turning point between *Assets* 1 and 2, as depicted in Figure 11.

Figure 10

Alternatively, if there does not exist any portfolio of *Assets* 1 and 2 that has higher expected return and lower variance than *Asset* 2, then there are no gains from diversification. The possible portfolios lie on a parabola that has no turning point between Assets 1 and 2, as depicted in Figure 10.
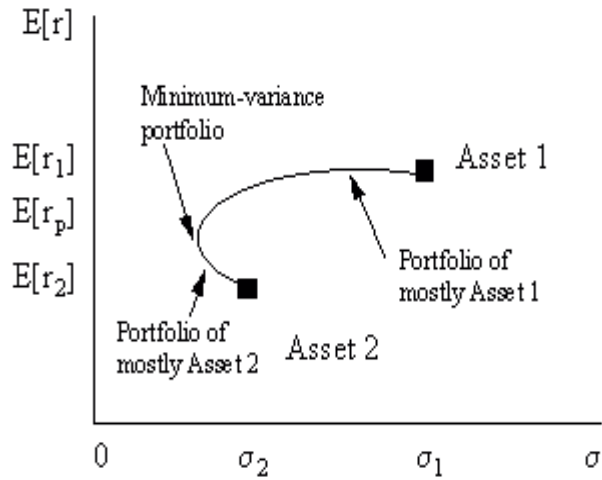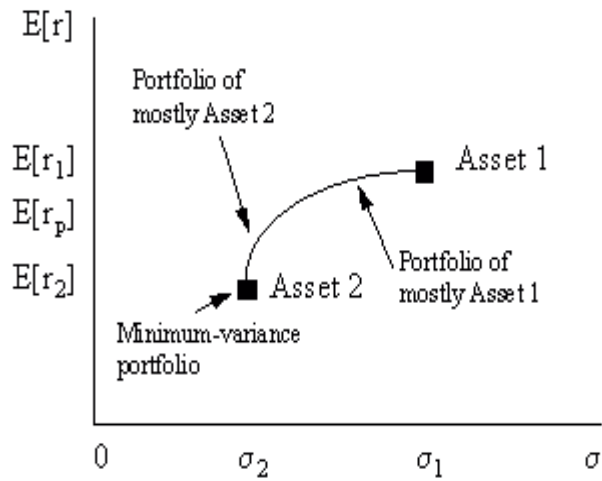

Figure 11

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Figures 10 shows different feasible combinations of risk and return, depending on the correlation between the assets. This graph in risk-return space (i.e., a graph where standard deviation is on one axis and expected returns on the other axis) it is also called the **mean variance frontier.**

4.9 The Minimum Variance Portfolio

To prove these claims, we find the portfolio that has the minimum variance among all the portfolios we can form by combining two assets. The minimum variance portfolio is found by solving:

$$\underset{w}{Min}\ \ \sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\ \sigma_1\sigma_2\rho_{1,2} \tag{18}$$

which has the first order condition

$$
\begin{aligned}
&2w^2\sigma_1^2 - 2\sigma_2^2 + 2w\sigma_2^2 + 2\rho_{1,2}\sigma_1\sigma_2 - 4w\rho_{1,2}\sigma_1\sigma_2 = 0 \\
&w\left(\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right) = \sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2 \\
&w = \frac{\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}
\end{aligned}
\tag{19}
$$

The minimum variance portfolio gives us an idea about the maximum benefits from diversification we can achieve by combining different assets. Now the portfolio will have less risk than *Asset* 2 alone whenever w>0, i. e., the minimum variance portfolio has a positive investment in the higher risk asset:

$$\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2 > 0 \quad\ or \quad \rho_{1,2} < \left(\frac{\sigma_2}{\sigma_1}\right) \tag{20}$$

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Condition (20) is precisely the condition we need in order to distinguish between the two scenarios in case 3 above: if (20) is satisfied, then there are gains from diversification and the minimum variance portfolio has positive investments in both assets. If the minimum variance portfolio has w<0, then we have to short sell one of the securities, i.e. we borrow one security in order to invest more in the other one.

We have defined the *mean-variance frontier*. After plotting all the portfolio combinations, the points farthest to the left represent the minimum variance portfolio. Consumers will only care about a certain portion of the minimum variance frontier - the portion with a positive slope. The negatively sloped part of the frontier implies a lower return for greater standard deviation. Our investors will not buy that trade-off. The positively sloped portion is called the efficient frontier. Portfolios on this frontier are referred to as *mean-variance efficient*. These portfolios maximize the expected return on the portfolio for a given variance.

These properties are reflected in all the portfolios on the efficient frontier. The opportunity set for investors follows. Because of our assumptions about investors, only the positively sloping portion of the minimum variance curve is held. The solid line represents the efficient frontier. Note that we have also included arrows representing the direction of the investors' preferences (more return and less risk).
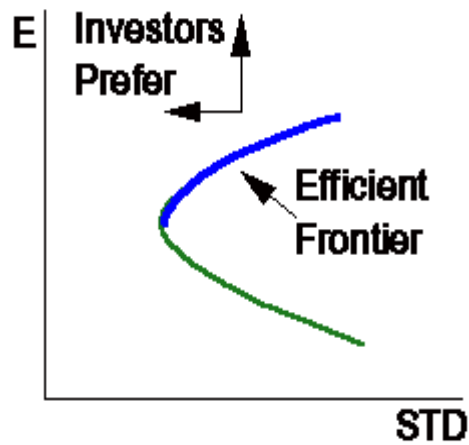
Figure 12

Example 8

Suppose *Assets* 1 and 2 have expected returns and standard deviations as follows:

Asset	Expected Return	Standard Deviation
1	20%	20%
2	10%	16%

Also assume that the returns of the two securities are perfectly negatively correlated $(\rho = -1)$. What is the composition of the minimum variance portfolio and what is its expected return and variance?

Here we use equation (16) $(\rho = -1)$ and set

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

Then the portfolio variance is zero. In this case,

$$w_1 = (16)/(20 + 16) = 0.44$$

which implies that 56% of the funds invested in the portfolio should be invested in *Asset* 2.

The expected return of this portfolio is:

$$E[r_P] = (0.44) \cdot 20 + (0.56) \cdot 10 = 14.44\%$$

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To check that the portfolio variance is indeed zero we calculate:

$$\sigma_p^2 = (0.44)^2 (20)^2 + (0.56)^2 (16)^2 + 2(0.44)(0.56)(-1)(20)(16) = 0$$

Example 9

Suppose that the correlation between the returns of the two assets from the previous example is $(\rho_{1,2} = 1)$.

What is the minimum variance portfolio that can be formed from *assets* 1 and 2 and what is the expected return and standard deviation of the 50/50 portfolio?

Finding the minimum variance portfolio is straightforward - since the two assets are perfectly positively correlated, there are no gains from diversification and the minimum variance portfolio will be the portfolio that puts all the weight on *Asset* 2 (which has the lower variance) and zero weight on *Asset* 1. This portfolio amounts to holding *Asset* 2 by itself and yields an expected return of 10% and has a standard deviation of 16%.

A portfolio of equal weights of Asset 1 and Asset 2 has expected return of

$$E[r_p] = (0.50) \cdot 20 + (0.50) \cdot 10 = 15.00\%$$

The variance of this portfolio is given by

$$(0.5)^2 (20)^2 + (0.5)^2 (16)^2 + 2(0.5)(0.5)(1)(20)(16) = 324$$

so the standard deviation is 18%.

Example 10

Suppose that the correlation between the returns of the two assets from the previous example is $(\rho_{1,2} = 0.5)$.

What is the expected return and standard deviation of a portfolio with equal weights in each security? And what is the composition of the minimum variance portfolio?

The expected return of the portfolio is given by:

$$E[r_p] = (0.50) \cdot 20 + (0.50) \cdot 10 = 15.00\%$$

The variance of the portfolio can be calculated as

$$(0.5)^2 (20)^2 + (0.5)^2 (16)^2 + 2(0.5)(0.5)(0.5)(20)(16) = 244$$

So, the standard deviation is 15.62%

Recall that *Asset* 2 has an expected return of 10% and a standard deviation of 16%. Clearly this portfolio of equal weights in *Asset* 1 and *Asset* 2 is preferred to holding *Asset* 2 by itself since the portfolio has an expected return of 15% and a standard deviation of 15.62%.

One way to find the minimum variance portfolio is to allow the weight to vary between 0 and 1 (from no investment in Asset 1 to all of our wealth in Asset 1) and examine the resulting portfolio. The expected returns and standard deviation of rates of return are as follows:

Weight (Asset 1)	Weight (Asset 2)	Expected Return	Standard Deviation
0.00	1.00	10.00	16.00
0.10	0.90	11.00	15.50
0.20	0.80	12.00	15.20
0.30	0.70	13.00	15.12
0.40	0.60	14.00	15.26
0.50	0.50	15.00	15.62
0.60	0.40	16.00	16.18
0.70	0.30	17.00	16.92
0.80	0.20	18.00	17.82
0.90	0.10	19.00	18.85
1.00	0.00	20.00	20.00

From this table we conclude that the minimum variance portfolio is given by setting w_1 equal to (roughly) 0.3, that is investing 30% in *Asset* 1 and 70% in *Asset* 2.

An alternative way to find the minimum variance portfolio is to use the result established above. Therefore, the minimum variance portfolio in this case is

$$w_1 = \frac{0.16^2 - 0.5 \cdot 0.20 \cdot 0.16}{0.20^2 + 0.16^2 - 2 \cdot 0.5 \cdot 0.20 \cdot 0.16} = 28.57\%$$

With 28.57 percent of our wealth in *Asset* 1 and 71.43 percent of our wealth in *Asset* 2, the expected portfolio return is 12.86% and the standard deviation of the portfolio return is 15.12%.

4.10 Risk Aversion and Portfolio Choice

We have talked about efficient portfolios being determined by the investors' preferences for more return to less return and less risk to more risk. Now let's consider individual investors. All investors are assumed to be risk averse and like more to less. On the following figure the efficient frontier is drawn and a set of indifference curves for an investor. The optimal portfolio for this particular investor is at the point of tangency between the indifference curve and the efficient frontier. [**Note** the indifference curves can never be tangent to the inefficient portion of the mean-variance frontier].
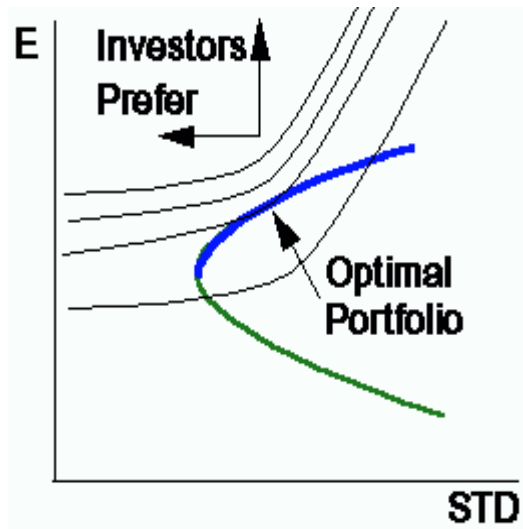
Figure 13

This is the optimal portfolio for one particular individual. Another person may be less tolerant of risk or very risk averse. This person's indifference curves are drawn below. Note that the optimal portfolio for this person has the smallest possible standard deviation.
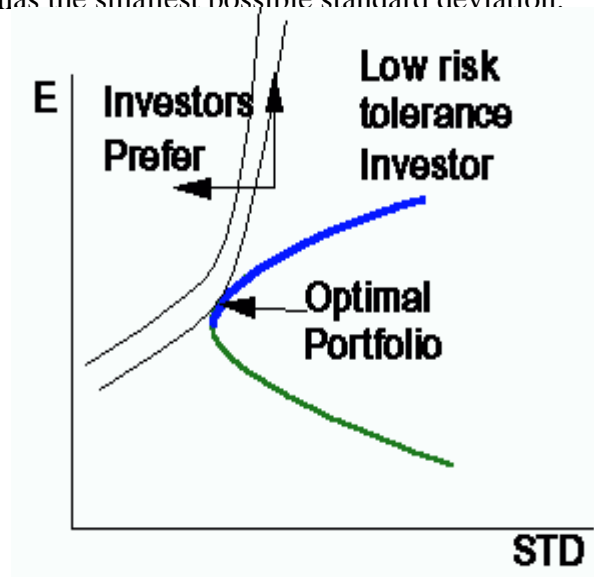
Figure 14

The next graph shows a set of indifference curves for a person that has high risk tolerance or low risk aversion. Note that this does not mean that the person is a risk lover. The utility function is still concave (risk aversion) but it is close to linear for very low risk aversion. The tangency point

28

on the indifference map shows this person choosing an efficient portfolio that has a large standard deviation compared to the other portfolios but note that the expected return is also higher.
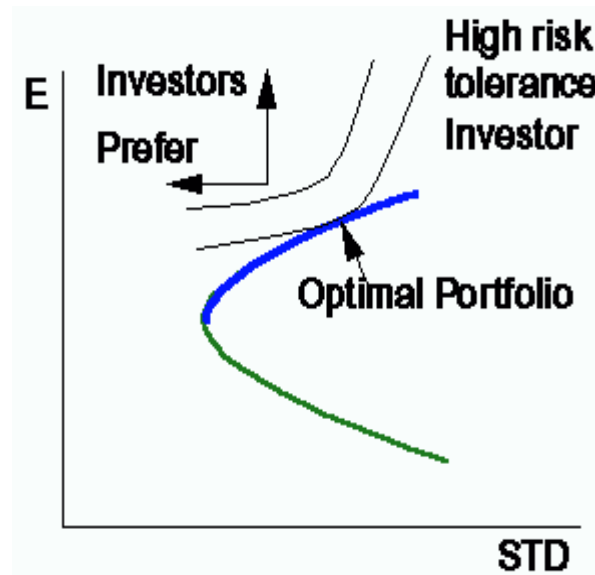
Figure 15

4.11 Mean-Variance Geometry with a Risk Free Security

Now consider the introduction of a riskless security like a Treasury bill. Suppose we invest in a combination of a portfolio on the efficient set (derived without the riskless security) and the riskless security. We can calculate the expected return and the standard deviation on this new portfolio.

$$E[r_p] = (1 - w)r_f + wE_e \qquad (21)$$

$$\sigma_p = w \cdot \sigma_e \qquad (22)$$

Where r_f represents the risk-free security, E_e represents the expected return of the portfolio on the efficient frontier and w is the proportion of funds invested in the risky security.
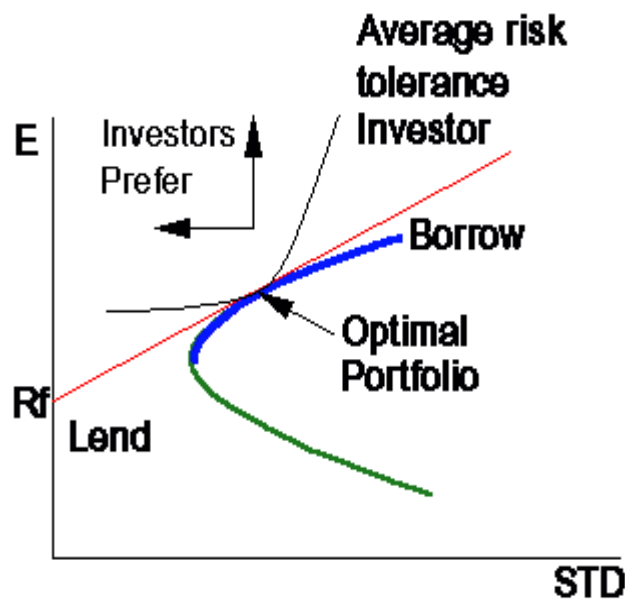
Figure 16

It is clear that the only portfolio on the old efficient frontier that is desirable is the tangency portfolio. If we chose another portfolio like A or B, this opportunity set is not efficient because for a given variance, you do not maximize the expected return.
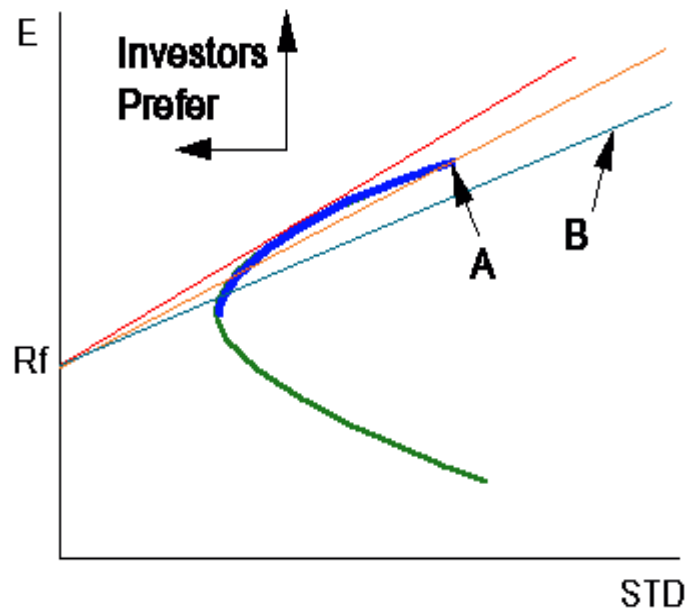
Figure 17

By introducing a new security, the risk-free security, we have to redraw the efficient set. It turns out that the new efficient set is the straight line from the risk-free rate to the tangency portfolio

and beyond. If you are on the line to the right of the tangency portfolio, you are borrowing at the

risk-free rate (note we are assuming that borrowing and lending rates are the same) and investing

in the tangency portfolio. If you are at the point R_f on the y-axis, this means that you have none

of your money ($w = 0$) in the risky asset. If you are at the point M, then all of your money is in

the portfolio of risky assets ($w = 1$) - which is the tangency portfolio.

So with a risky asset, there is only one optimal combination of risky investments for each

investor. The diagrams below show the optimal portfolio choice for individuals of average, low

and high risk tolerance. The optimal portfolio is the tangency of the indifference curve to the

efficient set.
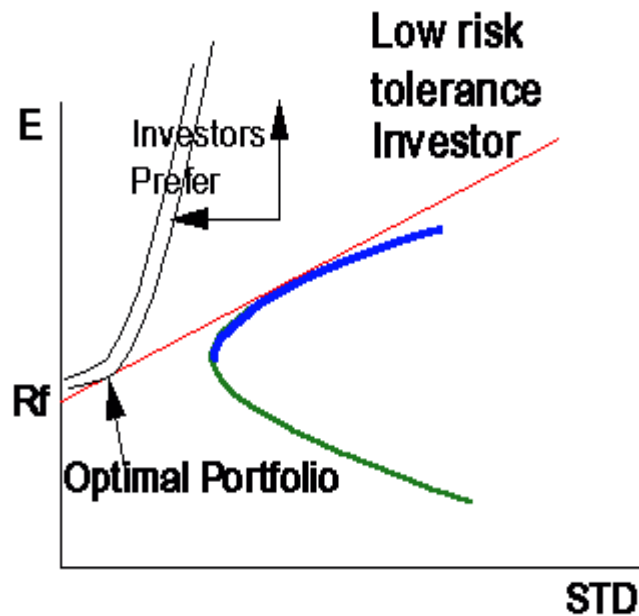
Figure 18
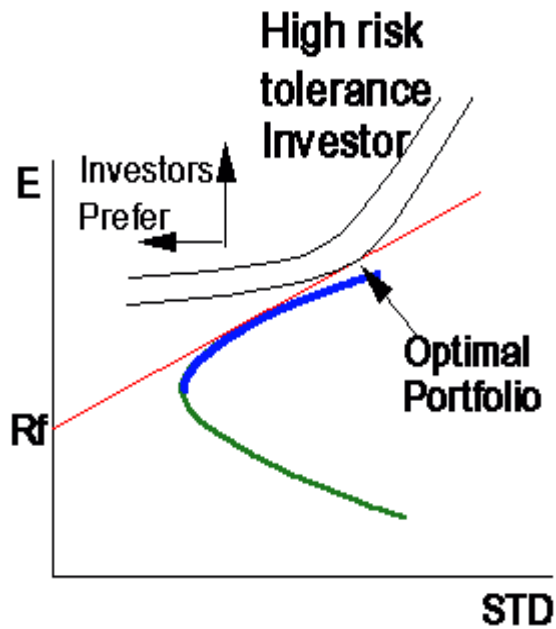
Figure 19

4.12 The Sharpe Measure

Consider investment in a riskless asset and calculate the portfolio mean and standard deviation.

We know from the standard deviation equation:

$$w = \frac{\sigma_p}{\sigma_e} \qquad (23)$$

Substitute this formula for w into the expected return equation.

$$E[r_p] = r_f + \frac{E_e - r_f}{\sigma_e}\sigma_p \qquad (24)$$

The term

$$\frac{E_e - r_f}{\sigma_e} \qquad (25)$$

is called the *Sharpe Measure* [Named after [William Sharpe] It is used to evaluate investments. Below is a graph depicting the expected return-standard deviation space. The Sharpe measure is the slope of the line from r_f (rise is $(E-r_f)$ over run which is STD). The intercept is the risk-free rate, r_f.
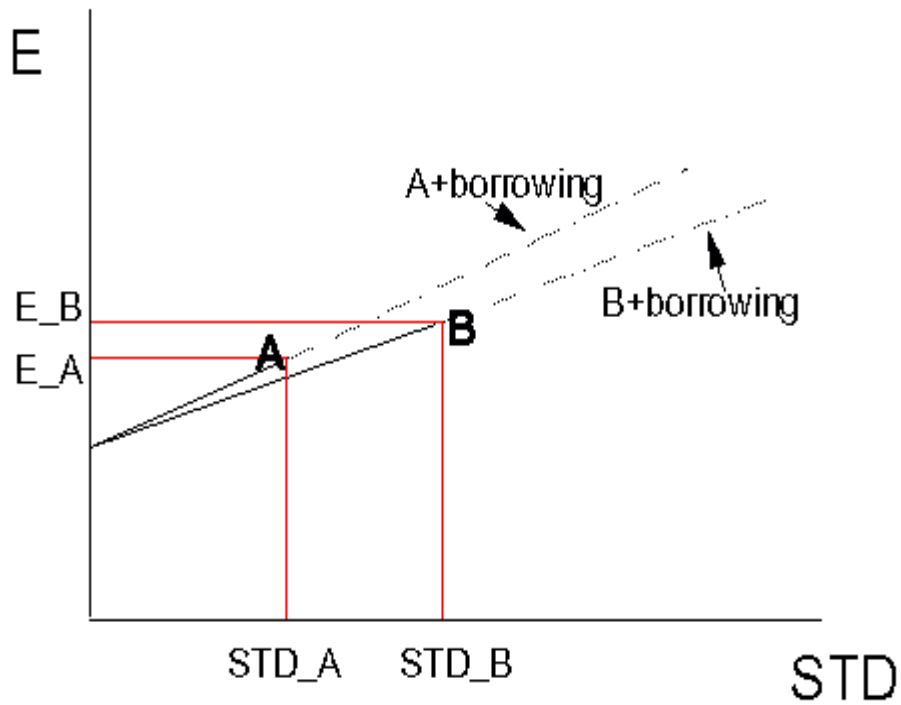
Figure 20

The higher the Sharpe measure is the better the security. On the graph we could combine a strategy of borrowing and buying portfolio A to achieve the same expected return as portfolio B with a much smaller variance.

Example 11

Let's consider a specific example. Suppose:

$r_f = 0.08$

$E_A = 0.20$

$\sigma_A = 0.20$

$E_B = 0.25$

$\sigma_B = 0.30$

Just looking at portfolio A and B it is unclear which is the best investment. B has the higher return - but it also has a higher variance. Let's first consider the Sharpe measures:

$$S_A = \frac{E_A - r_f}{\sigma_A} = \frac{0.20 - 0.08}{0.20} = 0.60$$

$$S_B = \frac{E_B - r_f}{\sigma_B} = \frac{0.25 - 0.08}{0.30} = 0.56$$

The measure suggests that portfolio B is dominated by a strategy of borrowing and holding portfolio A. Let's check this out by calculating the standard deviation of a levered portfolio of A that has exactly the same expected return as B:

$$0.25 = w(0.20) + (1 - w)0.08$$

Solving for weight w:

$$w = \frac{0.17}{0.12} = 1.4167$$

This suggests that a strategy of investing 141.67% of your money in A and borrowing 41.67% at the rate of $r_f = 8\%$ will deliver a portfolio return of 25% which is exactly the portfolio return for B. Now lets check the standard deviation of this levered portfolio:

$$\sigma_p = \sqrt{(1.4167)^2 \cdot (0.20)^2} = 28.33\%$$

Note that the other terms in the portfolio variance drop out because the variance of the risk-free asset is zero. We are left with a portfolio standard deviation of 28.33% which is lower than the 30% for portfolio B. The levered portfolio that contains A has the same

mean as B but a lower standard deviation. As a result, the levered portfolio with A is preferred to the investment in B.

We can expand the analysis to include all assets available in the market. We have argued that only the positively sloped portion of the minimum variance frontier of risky assets satisfied our portfolio selection rules. We can use the tools that we developed above to discriminate among the portfolios on the efficient frontier of risky assets. We will search for a combination of the risk-free asset and some risky portfolio that delivers the highest Sharpe measure. We know that the Sharpe measure is just the slope of the line that is drawn from the risk-free rate on the expected return axis. The portfolio with the highest Sharpe measure is the tangency portfolio.

So the best possible mean and standard deviation combinations are from the riskless and tangency portfolio. If 100% of your wealth is invested in the riskless asset, then your return is r_f and the standard deviation is zero. If 50% of your wealth is invested in the riskless asset and 50% of your wealth is in the tangency portfolio, then your portfolio lies in the middle between r_f and M on the straight line. If 100% of your money is in the tangency portfolio, the your expected return is the expected return on the tangency portfolio and your standard deviation is the standard deviation on the tangency portfolio. Finally, if you borrow money at the riskless rate and combine your borrowing with your initial wealth to buy the tangency portfolio, then your portfolio is to the right of M on the straight line. This straight line is called the ***Capital Market Line (CML).***
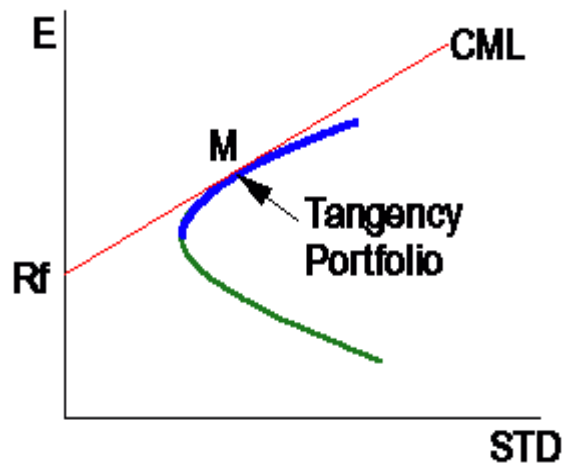
Figure 21

4.13 Common Sense Diversification

Whereas the benefits of diversification can be achieved through random selection of a number of stocks, a number of common sense procedures can be usefully employed to construct a diversified portfolio. For example:

- Diversify across industries: Investing in a number of different stocks within the same industry does not generate a diversified portfolio since the returns of firms within an industry tend to be highly correlated. Selecting stocks from different industries can increase diversification benefits.

- Diversify across industry groups: Some industries themselves can be highly correlated with other industries and hence diversification benefits can be maximized by selecting stocks from those industries that tend to move in opposite directions or have very little correlation with each other.

- Diversify across geographical regions: Companies whose operations are in the same geographical region are subject to the same risks in terms of natural disasters and state or local tax changes. Investing in companies whose operations are not in the same geographical region can diversify these risks.

- Diversify across economies: Stocks in the same country tend to be more highly correlated than stocks across different countries. This is because many taxation and regulatory issues apply to all stocks in a particular country. International diversification provides a means for diversifying these risks.

- Diversify across asset classes: Investing across asset classes such as stocks, bonds, and real property also produces diversification benefits. The returns of two stocks tend to be more highly correlated, on average, than the returns of a stock and a bond or a stock and an investment in real estate.

Appendix

Example 12

Suppose Assets 1 and 2 have expected returns and standard deviations as follows:

Security	Expected Return	Standard Deviation
1	20%	20%
2	10%	16%

Furthermore, suppose that the returns of the two securities are perfectly negatively correlated with $(\rho = -1)$.

What is the expected return and standard deviation of a portfolio with equal weights in each security?

Since the weight of each asset is $w_1 = w_2 = 0.5$, we find that

$$E[r_p] = 0.5 \cdot (20\%) + 0.5 \cdot (10\%) = 15\%$$

We compute the variance of the returns of the portfolio by plugging in the values for our 50/50 portfolio:

$$\sigma_p = (0.5^2)(0.20^2) + (0.5^2)(0.16^2) + 2(0.5)(0.5)(0.20)(0.16)(-1) = 0.0004$$

in which case the standard deviation of the returns of the portfolio is $0.02 = 2\%$.

Recall that Asset 2 has an expected return of 10% and a standard deviation of 16%. Clearly this portfolio of equal weights in Asset 1 and Asset 2 is preferred to holding Asset 2 by itself since the portfolio has an expected return of 15% and a standard deviation of 2%.

A.1 The *N* Asset Case

For an *N* asset portfolio, the portfolio return is just the sum of the asset returns times the weights each of the assets has in the portfolio. This is just:

$$r_p = \sum_i w_i r_i \qquad\qquad\qquad\qquad\qquad\qquad (A\ 1)$$

w is a vector containing the respective weights of the N assets. $w = (w_1, w_2, ..., w_N)$ That is, if we have \$1 million to invest and we place \$100,000 in security i, then $w_i = 0.10$. The weights must all sum to one. This means that all money must be allocated. We define r as a vector containing the returns of the N assets: $r = (r_1, r_2, ..., r_N)$. Then we can express (A 1) in matrix notation as:[23]

$$r_p = w'r \qquad\qquad\qquad\qquad\qquad\qquad (A\ 2)$$

The portfolio expected return is just the expected asset returns times the weights each of the assets has in the portfolio. The portfolio expected return is just the sum of the expected asset returns times the weights each of the assets has in the portfolio. Let **E** be a vector containing the expected returns of the assets:

$$E' = (E[r_1], E[r_2], ..., E[r_N]) \qquad\qquad\qquad\qquad\qquad\qquad (A\ 3)$$

so the portfolio's expected return can also be defined in terms of sums and matrix multiplication:

.

$$E_p = \sum w_i E(r_i) = w'E \qquad\qquad\qquad\qquad\qquad\qquad (A\ 4)$$

Note that the above formula expands to the formula above for the two asset case. The variance of the portfolio return is a little more complicated when there are more than two assets. It will still be a function of the weights, variances and covariances, but it is harder to express as a simple formula:

---

[2]   This product can be calculated in *Excel* using the *sumproduct* function.

.

$$\sigma_p^2 = \sum_i \sum_j \sigma_i \sigma_j \rho_{ij} w_i w_j \tag{A 5}$$

Let **V** be the variance-covariance matrix (variances along diagonal and covariances off the diagonal):

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_N^2 \end{pmatrix} \tag{A 6}$$

Then (A 5) becomes:

$$\sigma_P^2 = w\text{'}Vw \tag{A 7}$$

For the two asset case.

$$\sigma_p^2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1\sigma_1^2 + w_2\sigma_{2,1} & w_1\sigma_{1,2} + w_2\sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$= w_1^2\sigma_1^2 + w_1 w_2 \sigma_{2,1} + w_1 w_2 \sigma_{1,2} + w_2^2\sigma_2^2 \tag{A 8}$$
$$= w_1^2\sigma_1^2 + 2 w_1 w_2 \sigma_{1,2} + w_2^2\sigma_2^2$$

The matrix algebra for the *N* asset case can easily be implemented in *Excel*.

The covariance of two portfolio returns, each denoted by their own set of weights, say w_a, w_b can also be found using matrix algebra:

---

[3]   Vectors are usually written as a column. The mark after the **E** says to take the transpose of the column, or, in other words, look at it as a row.

$$r_a = w_a'r$$
$$r_b = w_b'r$$ (A 9)
$$Cov(r_a, r_b) = w_a'Vw_b$$

Example 13

Consider forming a portfolio with three assets.

The expected returns are $E'(r_1) = 0.25$, $E'(r_2) = 0.19$, and $E'(r_3) = 0.12$. Hence:

$$E = \begin{pmatrix} E(r_1) \\ E(r_2) \\ E(r_3) \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.19 \\ 0.12 \end{pmatrix}$$

The variance-covariance matrix is V

$$V = \begin{vmatrix} 0.090 & 0.030 & 0.007 \\ 0.030 & 0.100 & -0.020 \\ 0.007 & -0.020 & 0.010 \end{vmatrix}$$

Portfolio 1 weights are $w' = (.3, .4, .3)$

Portfolio 2 weights are $w' = (.1, .6, .3)$

1. Calculate the standard deviations for each asset's return.

2. Calculate the correlation between the asset's returns.

3. Calculate the portfolio variances.

4. Calculate the covariance between the portfolios.

Recall that the diagonal elements of the Variance/Covariance matrix contains the variances. The standard deviations can be found by taking the square roots of those variances.

$$(0.090)^{0.5} = 0.300$$
$$(0.100)^{0.5} = 0.316$$
$$(0.010)^{0.5} = 0.100$$

The correlation coefficient can be calculated using the following relationship:

$$\rho_{i,j} = \frac{\sigma_{i,j}}{\sigma_i \cdot \sigma_j}$$

The covariances can be found in the V matrix where the covariances between assets j and k will be in *row j*, *column k* (and also in *column j*, *row k*).

$$\sigma_{1,2} = 0.030$$
$$\sigma_{1,3} = 0.007$$
$$\sigma_{2,3} = -0.020$$

Solving for the correlation coefficients:

$$\rho_{1,2} = \frac{0.030}{(0.030 \cdot 0.316)} = 0.316$$

$$\rho_{1,3} = \frac{0.007}{(0.300 \cdot 0.100)} = 0.230$$

$$\rho_{2,3} = \frac{-0.020}{(0.316 \cdot 0.100)} = -0.630$$

Solving for the portfolio variances is easy using (A 7).[4] The solution for the first portfolio is:

$$\sigma_{p1}^2 = (0.3, 0.4, 0.3) \cdot \begin{vmatrix} 0.090 & 0.030 & 0.007 \\ 0.030 & 0.100 & -0.020 \\ 0.007 & -0.020 & 0.010 \end{vmatrix} \cdot \begin{vmatrix} 0.3 \\ 0.4 \\ 0.3 \end{vmatrix} = 0.02866$$

The solution for the second portfolio is:

$$\sigma_{p2}^2 = (0.1, 0.6, 0.3) \cdot \begin{vmatrix} 0.090 & 0.030 & 0.007 \\ 0.030 & 0.100 & -0.020 \\ 0.007 & -0.020 & 0.010 \end{vmatrix} \cdot \begin{vmatrix} 0.3 \\ 0.1 \\ 0.6 \end{vmatrix} = 0.03462$$

We can calculate:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2}$$

to find the covariance between these two portfolios. We could treat each of these portfolios as a separate asset, and create a new portfolio using equal weights. The variance of the new portfolio can be found, because it is just a portfolio of our three original assets. The variances of the two portfolios can be plugged in to the right hand side of the equation. That leaves the covariance between the two portfolios as the only unknown. Solving for that, we find that the covariance between the two portfolios is 0.029040.

Example 14

We will now consider an example of the effects of diversification. Previously, we combined securities and looked at the effect on the portfolio variance for different correlation coefficients between the securities. We found that using equal weights in the two portfolios, a lower correlation coefficient led to lower portfolio variance. In this example, we will look at a given correlation and vary the portfolio weights to trace the effect on the portfolio variance.

The example comes from a classic article by Bodie and Rosansky, "*Risk and Return in Commodity Futures*" which was published in the *Financial Analysts Journal* in 1980. We will trace out the return and standard deviation of a portfolio of common stocks and futures. A number of tables are presented.

The tables appended below give the data for this example. Table 1 and Table 2 document the cumulative wealth relatives and the year by year rates of return of five portfolios from 1949 to 1976. Table 3 provides mean returns and standard deviations for these portfolios.

Table 4 gives the correlation matrix for the different returns. Table 5 provides returns on selected common stock-commodity futures portfolios.

We will be concerned with the common stocks and the commodity futures. Note that the common stocks have a -24% correlation with the commodity futures. Previously, we showed that combining two portfolios with a zero correlation reduced the variance of the portfolio. This was referred to as diversification. The common stocks and futures have negative correlation. This suggests that holding both in a portfolio will produce a portfolio variance that is less than the variance of the individual components.

We are given that for *Asset* 1, the common stocks:

$E[r_1] = 13.05\%$
$\sigma_1 = 18.95\%$

Also, *Asset* 2, the commodity futures:

$E[r_2] = 13.83\%$
$\sigma_2 = 22.43\%$

We are also given the correlation coefficient and the variances can easily be calculated:

$\rho_{1,2} = -0.24$
$\sigma_1^2 = 0.0359$
$\sigma_2^2 = 0.0503$

The next step is to calculate the portfolio mean and standard deviation for various weights.

Portfolio 1 ($w_1 = 1$ $w_2 = 0$)

The mean return is:

$E[r_{p1}] = 1 \cdot (13.05) + 0 \cdot (13.83) = 13.05\%$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(1 \cdot 0.0359) + (0 \cdot 0.0503) + (2 \cdot 1 \cdot 0 \cdot (-0.24)) \cdot 0.895 \cdot 0.2243}$$
$$= \sqrt{0.0359} = 0.1895 = 18.95\%$$

Portfolio 2 ($w_1 = 0.8$ $w_2 = 0.2$)

The mean return is:

$$E[r_{p2}] = 0.8 \cdot (13.05) + 0.2 \cdot (13.83) = 13.21\%$$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(0.64 \cdot 0.0359) + (0.04 \cdot 0.0503) + (2 \cdot 0.8 \cdot 0.2 \cdot (-0.24)) \cdot 0.1895 \cdot 0.2243}$$
$$= \sqrt{0.0217} = 0.1474 = 14.74\%$$

Portfolio 3 ($w_1 = 0.6$ $w_2 = 0.4$)

The mean return is:

$$E[r_{p3}] = 0.6 \cdot (13.05) + 0.4 \cdot (13.83) = 13.36\%$$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(0.36 \cdot 0.0359) + (0.16 \cdot 0.0503) + (2 \cdot 0.6 \cdot 0.4 \cdot (-0.24)) \cdot 0.1895 \cdot 0.2243}$$
$$= \sqrt{0.0161} = 0.1268 = 12.68\%$$

Portfolio 4 ($w_1 = 0.4$ $w_2 = 0.6$)

The mean return is:

$$E[r_{p3}] = 0.4 \cdot (13.05) + 0.6 \cdot (13.83) = 13.52\%$$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(0.16 \cdot 0.0359) + (0.36 \cdot 0.0503) + (2 \cdot 0.4 \cdot 0.6 \cdot (-0.24)) \cdot 0.1895 \cdot 0.2243}$$
$$= \sqrt{0.0190} = 0.1377 = 13.77\%$$

Portfolio 5 ($w_1 = 0.2$ $w_2 = 0.8$)

The mean return is:

$$E[r_{p3}] = 0.2 \cdot (13.05) + 0.8 \cdot (13.83) = 13.67\%$$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(0.04 \cdot 0.0359) + (0.64 \cdot 0.0503) + (2 \cdot 0.2 \cdot 0.8 \cdot (-0.24)) \cdot 0.1895 \cdot 0.2243}$$
$$= \sqrt{0.0304} = 0.1743 = 17.43\%$$

Portfolio 6 ($w_1 = 0$ $w_2 = 1$)

The mean return is:

$$E[r_{p3}] = 0 \cdot (13.05) + 1 \cdot (13.83) = 13.83\%$$

The standard deviation is:

$$\sigma_{p1} = \sqrt{(0 \cdot 0.0359) + (1 \cdot 0.0503) + (2 \cdot 0 \cdot 1 \cdot (-0.24)) \cdot 0.1895 \cdot 0.2243}$$
$$= \sqrt{0.0503} = 0..2243 = 22.43\%$$

These calculations verify the numbers presented in Table 5.

The following graph shows the risk and return.

Mean-Standard Deviation Frontier for Common Stocks and Commodities
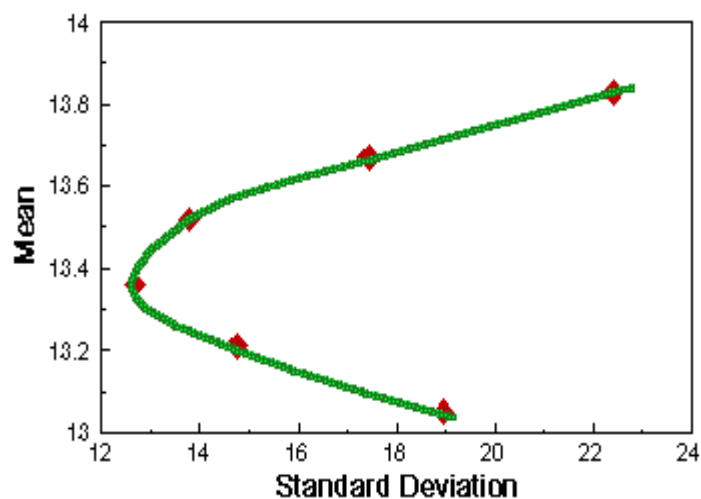
Note that with the negative correlation between the assets, the amount invested in the component securities has a large effect on the portfolio variance.

TABLE 1

Index of Year-End Cumulative Wealth Relatives, 1949-76

Year	Common Stocks	Commodity Futures	Number of Commodities	Long Term Government Bonds	US Treasury Bills	Consumer Price Index
1949	1.000	1.000	-	1.000	1.000	1.000
1950	1.317	1.526	10	1.000	1.012	1.058
1951	1.633	1.934	10	0.51	1.027	1.121
1952	1.933	1.910	10	0.972	1.044	1.131
1953	1.914	1.789	10	1.007	1.063	1.137
1954	2.922	2.056	13	1.080	1.073	1.132
1955	3.844	1.957	13	1.066	1.089	1.136
1956	4.096	2.246	13	1.006	1.116	1.168
1957	3.654	2.193	13	1.081	1.151	1.204
1958	5.239	2.164	15	1.016	1.169	1.225
1959	5.863	2.176	13	0.993	1.203	1.244
1960	5.892	2.174	13	1.129	1.235	1.262
1961	7.477	Z.208	13	1.140	1.262	1.270
1962	6.824	2.227	13	1.219	1.296	1.286
1963	8.380	2.735	13	1.213	1.337	1.307
1964	9.762	3.067	14	1.277	1.383	1.322
1965	10.977	3.393	16	1.286	1.438	1.348
1966	9.872	3.890	17	1.333	1.506	1.393
1967	12.239	4.050	19	1.210	1.570	1.436
1968	13.593	4.101	19	1.207	1.651	1.503
1969	12.437	4.955	21	1.146	1.760	1.595
1970	12.935	5.550	22	1.285	1.877	1.693
1971	14.787	5.734	23	1.454	1.957	1.739

1972	17.592	7.666	23	1.537	2.032	1.799
1973	15.013	15.45	23	1.520	2.174	1.957
1974	11.037	20.38	23	1.586	2.347	2.196
1975	15.144	19.57	23	1.732	2.483	2.350
1976	18.754	22.06	23	2.022	2.609	2.463

TABLE 2

Year-by-Year Rates of Return, 1950-76

Year	Common Stocks	Commodity Futures	Long Term Government Bonds	US Treasury Bills	Rate of Inflation
1950	31.71	52.61	0.06	1.20	5.79
1951	24.02	26.71	-3.94	1.49	5.87
1952	18.37	-1.16	1.66	1.66	0.88
1953	-0.99	-6.32	3.63	1.82	0.62
1954	52.62	14.88	7.19	0.86	-0.50
1955	31.36	-4.79	-1.30	1.57	0.37
1956	6.56	14.75	-5.59	2.46	2.86
1957	-10.78	-2.34	7.45	3.14	3.02
1958	43.36	-1.33	-6.10	1.54	1.76
1959	11.95	0.54	-2.26	2.95	1.50
1960	0.47	-0.09	13.78	2.66	1.48
1961	26.89	1.55	0.97	2.13	0.67
1962	-8.73	0.87	6.89	2.73	1.22
1963	22.80	22.84	1.21	3.12	1.65
1964	16.48	12.13	3.51	3.54	1.19
1965	12.45	10.62	0.71	3.93	1.92
1966	-10.06	14.65	3.65	4.76	3.35
1967	23.98	4.13	-9.19	4.21	3.04
1968	11.06	1.24	-0.26	5.21	4.72
1969	-8.50	20.84	-5.08	6.58	6.11
1970	4.01	11.99	12.10	6.53	5.49
1971	14.31	3.31	13.23	4.39	3.36
1972	18.98	33.71	5.68	3.84	3.41
1973	-14.66	101.54	-1.11	6.93	8.80
1974	-26.48	31.96	4.35	8.00	12.20
1975	37.20	-4.01	9.19	5.80	7.01
1976	23.84	12.75	16.75	5.08	4.81

TABLE 3

Annual Rates of Return on Alternative Investments, 1930-76 (Nominal Returns)

Series	Mean	Standard Deviation	Standard Error	# Years with Negative Returns	Mean Annual Loss[a]	Highest Annual Return (year)	Lowest Annual Return (year)
Common Stocks	13.05	18.95	3.65	7	11.46	52.62 (1954)	-26.48 (1974)
Commodity Futures with T-Bills	13.83	22.43	4.32	7	2.87	101.54 (1973)	-6.32 (1953)
Long Term Government Bonds	2.94	6.53	1.26	9	3.86	16.75 (1976)	-9.19 (1967)

						8.00 (1974)	0.86 (1954)
US T-Bills	2.63	1.95	0.38	0	-		
Rate of Inflation	3.43	2.90	0.56	1	-	12.20 (1974)	-0.50 (1954)

Annual Rates of Return on Alternative Investments, 1930-76 (Nominal Returns)

Series	Mean	Standard Deviation	Standard Error	# Years with Negative Returns	Mean Annual Loss[a]	Highest Annual Return (year)	Lowest Annual Return (year)
Common Stocks	9.58	19.65	3.78	9	13.11	53.37 (1954)	-34.79 (1974)
Commodity Futures with T-Bills	9.81	19.44	3.74	11	3.64	85.24 (1973)	-10.30 (1975)
Long Term Government Bonds	-0.31	6.81	1.31	12	6.55	12.11 (1960)	-11.90 (1967)
US T-Bills	0.22	1.80	0.35	7	2.41	2.32 (1964)	-4.39 (1950)

Annual Rates of Return on Alternative Investments, 1930-76 (Excess Returns[c])

Series	Mean	Standard Deviation	Standard Error	# Years with Negative Returns	Mean Annual Loss[a]	Highest Annual Return (year)	Lowest Annual Return (year)
Common Stocks	9.42	20.12	3.87	9	3.21	51.76 (1954)	-34.48 (1974)
Commodity Futures with T-Bills	9.77	21.39	4.12	13	3.91	91.59 (1973)	-10.05 (1975)
Long Term Government Bonds	-0.79	6.43	1.24	17	4.73	11.67 (1976)	-13.40 (1967)

a *The mean annual loss is defined as the sum of the annual losses (negative rates of return) divided by the number of years in which losses occurred.*

b *The real rate of return, R_r is defined by: $(1+R_n)/(1+i) -1$ where R_n is the nominal rate of return and i is the rate of inflation as measured by the proportional change in the Consumer Price Index.*

c *The excess return is the difference between the nominal rate of return and the Treasury bill rate*

TABLE 4

Correlation Matrix of Annual Rates of Return, 1950-76 (Nominal Returns)

	Commodity Futures	Long Term Government Bonds	Treasury Bills	Inflation
Common Stocks	-0.24	-0.10	-0.57	-0.43
Commodity Futures		-016	0.34	0.58
Long Term Government Bonds			0.21	0.03
Treasury Bills				0.76

Correlation Matrix of Annual Rates of Return, 1950-76 (Real Returns)

	Commodity Futures	Long Term Government Bonds	Treasury Bills	Inflation
Common Stocks	-0.25	-0.14	0.18	-0.54
Commodity Futures		-036	-0.48	0.48
Long Term Government Bonds			0.46	-0.38
Treasury Bills				-0.75

Correlation Matrix of Annual Rates of Return, 1950-76 (Excess Returns)

	Commodity Futures	Long Term Government Bonds	Inflation
Common Stocks	-0.20	-0.08	-0.48
Commodity Futures		-026	-0.52
Long Term Government Bonds			-0.20

TABLE 5

Nominal Rates of Return on Selected Common Stock-Commodity Futures Portfolios, 1950-76

Proportion of Portfolio Invested in Stocks

Year	100%	80%	60%	40%	20%	0%
1950	31.71%	35.89%	40.07%	44.25%	48.43%	52.61%
1951	24.2	24.56	25.10	25.64	26.17	26.71
1952	18.37	14.45	10.53	6.62	2.70	-1.22
1953	-0.99	-2.06	-3.12	-4.19	-5.26	-6.32
1954	52.62	45.07	37.52	29.98	22.43	14.88
1955	31.56	24.29	17.02	9.75	2.48	-4.79
1956	6.56	8.20	9.84	11.47	13.11	14.75
1957	-10.78	-9.09	-7.40	-5.72	-4.03	-2.34
1958	43.36	34.42	Z5.48	16.55	7.61	-1.33
1959	11.95	9.67	7.39	5.11	2.82	0.54
1960	0.47	0.36	0.24	0.13	0.02	-0.09
1961	26.89	21.82	16.75	11.68	6.62	1.55
1962	-8.73	-6.81	-4.89	-2.97	-1.05	0.87
1963	22.80	22.81	22.81	22.82	22.83	22.84
1964	16.48	15.61	14.75	13.87	13.00	12.13
1965	12.45	12.08	11.72	11.35	10.99	10.62
1966	-10.06	-5.12	-0.18	4.76	9.71	14.65
1967	23.98	20.01	16.04	12.07	8.10	4.13
1968	11.06	9.10	7.13	5.17	3.20	1.24
1969	-8.50	-2.63	3.24	9.11	14.98	20.84
1970	4.01	5.61	7.20	8.80	10.39	11.99
1971	14.31	17.11	9.91	7.71	5.31	3.31
1972	18.98	21.93	24.87	27.82	30.76	33.71
1973	-14.66	8.53	31.82	53.06	78.30	101.54
1974	-26.48	-14.79	-3.10	8.58	20.27	31.98
1975	37.20	28.96	20.72	12.47	4.Z3	-4.01
1976	23.84	21.62	19.40	17.18	14.96	12.75
Mean	**13.05**	**13.21**	**13.36**	**13.52**	**13.67**	**13.83**
Std Dev	**18.95**	**14.74**	**12.68**	**13.77**	**17.43**	**22.43**