TEACHING NOTE 98-01:

CLOSED-FORM AMERICAN CALL OPTION PRICING:

ROLL-GESKE-WHALEY

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It is well-known that an American call option on a stock that pays no dividends during the life of the option will not be exercised early and, hence, can be valued as a European option with the standard Black-Scholes formula. If the underlying stock pays a dividend during the life of the option, early exercise could possibly be optimal, thereby, giving the American call a premium over a European call and rendering the Black-Scholes model inappropriate.[1] Valuation of such an option can be done by numerical methods, such as the binomial model, but it is also possible to obtain a closed-form valuation model. For a stock paying a single dividend during the life of the option, the model is called the Roll-Geske-Whaley model after Roll (1977), Geske (1981), and Whaley (1981). The model is based on Geske's (1979) compound option model.

We require only that there is but one known dividend paid over the option's life. The amount of the dividend is d and the time in years until the ex-dividend date is t.[2] We assume that as the stock goes ex-dividend, the stock price falls by the amount of the dividend, although this assumption can be relaxed. We are also given the time to expiration in years, T, stock volatility, σ, and the continuously compounded risk-free rate, r. Let us first define the following value $S - de^{-rt}$, which is simply the stock price minus the present value of the dividend. We assume that this adjusted stock price follows the standard lognormal diffusion that is typically used in modeling asset price

[1]It is well-known that the lower bound of an American call on a stock with dividends is S - PV(X) - PV(D) where S is the current stock price, PV(X) is the present value of the exercise price and PV(D) is the present value of the dividends over the life of the option. If this option were exercised, it would give a payoff of S - X. Clearly if X - PV(X) > PV(D) early exercise will not be justified. This will occur with a combination of small dividends, a high interest rate, and a high exercise price. Such an option can, therefore, be valued using the Black-Scholes model with the stock price reduced by the present value of the dividends.

[2]For example if the ex-dividend day is 35 days from now, t = 35/365.

dynamics.[3] We denote the price of a European call on this stock as c(S,X,T,d,t) and the price of an American call on this stock as C(S,X,T,d,t).

We can replicate the American call with the following combination of options:

(a) A long position in a European call with time to expiration T and exercise price X.

(b) A short position in a European compound call option where the underlying option is the European call in (a). This compound option has a time to expiration of t and an exercise price of $S_t^* + d - X$, where S_t^* is defined below.

(c) A long position in a European call with time to expiration of t and exercise price of S_t^*.

The value S_t^* is the critical ex-dividend stock price above which the American call would be exercised the instant before the stock goes ex-dividend. Since it is an ex-dividend price, $S_t^* = S_t - d$. Suppose that the option is not exercised and the stock goes ex-dividend. Then, with no dividends remaining over the life of the option, the option is a standard European call on a stock with no dividends and is easily valued by the Black-Scholes model with remaining time to expiration of T - t. If the option were exercised, it would pay off $S_t + d - X$ where S_t is the ex-dividend stock price. The option holder would have a stock worth S_t, a dividend of d, but have paid out X. The critical ex-dividend stock price for justification of exercise is the one such that $c(S_t^*,X,T-t) = S_t^* + d - X$. This value must be derived iteratively by plugging in values into the Black-Scholes model with time to expiration of T - t until the option price equals the $S_t^* + d - X$. A good starting point estimate of S_t^* is X - d since the option would have to give a positive payoff to justify exercise, but the actual value of S_t^* is likely to be much higher. Standard iterative equation-solving techniques like Newton-Raphson can speed up the search. If early exercise is not justified at any price, due to too small of a dividend relative to the other values in the model, S_t^* will be infinite. In that case our final solution will converge to Black-Scholes.

Now let us see what happens to each of the three options at the ex-dividend instant.

$S_t < S_t^*$ (ex-dividend price is less than the critical price):

[3]Roll assumed that the full stock price followed the lognormal diffusion, but this cannot be true if the dividend component of it is non-stochastic. This correction was made by Whaley.

(a) This is a European call and cannot be exercised until its expiration T. It is simply worth $c(S_t, X, T-t)$ when the stock goes ex-dividend.

(b) This compound option will be exercised or not depending on whether the value of the option in (a) exceeds the compound option's exercise price $S_t^* + d - X$. Since the actual stock price S_t is less than S_t^* in this case, then $c(S_t, X, T-t) < c(S_t^*, X, T-t)$, but by definition $c(S_t^*, X, T-t) = S_t^* + d - X$. Thus $c(S_t, X, T-t) < S_t^* + d - X$. Consequently, the compound option is out-of-the-money and is not exercised.

(c) This is an expiring European call. Its exercise price S_t^* exceeds the ex-dividend stock price. So it expires worthless.

Thus if $S_t \geq S_t^*$, this combination of options produces no cash flow at this point and leaves us holding a European call with remaining expiration T - t.

$S_t \geq S_t^*$ (ex-dividend price is greater than the critical price):

(a) This call will be worth $c(S_t, X, T-t)$ when the stock goes ex-dividend.

(b) The compound option is expiring. Using the arguments presented above for (b), this option is now in-the-money and is exercised. Since we are short this option, we deliver the option in (a) and receive the exercise price $S_t^* + d - X$.

(c) This option is a European call expiring right now. Its exercise price S_t^* is less than the current stock price S_t. So it expires in-the-money. We own this option so we pay S_t^* and receive stock worth S_t.

Thus if $S_t \geq S_t^*$ the overall cash flow is $S_t - S_t^* + S_t^* + d - X = S_t + d - X$. Thus, we paid out X and received stock worth S_t and its dividend d.

It should be apparent that these cash flows are the same as those of an American call. We can easily find the value of the American call by adding the values of options (a) and (c), which are given by the Black-Scholes model, and subtracting the value of option (b), which as a compound option is given by Geske's compound option formula. This can be written as

$$
\begin{aligned}
&(a)\ \ (S - de^{-rt})N_1(a_1) - Xe^{-rT}N_1(a_2)\\
&(b)\ \ (S - de^{-rt})N_2(b_1,a_1;\rho) - Xe^{-rT}N_2(b_2,a_2;\rho) - (S_t^* + d - X)e^{-rt}N_1(b_2)\\
&(c)\ \ (S - de^{-rt})N_1(b_1) - S_t^* e^{-rt}N_1(b_2),
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{\ln((S - de^{-rt})/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\\
a_2 &= a_1 - \sigma\sqrt{T}\\
b_1 &= \frac{\ln((S - de^{-rt})/S_t^*) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}\\
b_2 &= b_1 - \sigma\sqrt{t}\\
\rho &= -\sqrt{t/T}
\end{aligned}
$$

where $N_1(.)$ is the univariate normal probability and $N_2(.)$ is the bivariate normal probability. These formulas can be consolidated to equal

$$
\begin{aligned}
&(S - de^{-rt})N_1(a_1) - (X - d)e^{-rt}N_1(b_2)\\
&(S - de^{-rt})[N_1(b_1) - N_2(b_1,a_1;\rho)]\\
&\quad - Xe^{-rT}[N_1(a_2) - N_2(b_2,a_2;\rho)].
\end{aligned}
$$

The following relationships exist between the univariate and the bivariate normal distributions.

$$N_1(x) - N_2(x,y;\rho) = N_2(x,-y;-\rho)$$

$$N_2(x,y;\rho) = N(y,x;\rho).$$

Therefore, the second expression in the equation directly above can be written as

$$(S - de^{-rt})N_2(b_1,-a_1;-\rho),$$

and the third expression can be written as

$$Xe^{-rT}[N_1(a_2) - N_2(a_2,b_2;\rho)] = Xe^{-rT}N_2(a_2,-b_2;-\rho).$$

Thus, our formula can now be written as

$$(S - de^{-rt})N_1(a_1) + (S - de^{-rt})N_2(b_1,-a_1;-\rho) - Xe^{-rT}N_2(a_2;-b_2;-\rho) - (X - d_1)e^{-rt}N_1(b_2).$$

Now we can use the following relationships:

$$N_1(a_1) + N_2(b_1,-a_1;-\rho) = N_1(b_1) + N_2(a_1,-b_1;-\rho)$$

$$N_1(a_1) + N_1(b_1) - N_2(a_1,b_1;\rho) = N_1(b_1) + N_2(a_1,-b_1;-\rho),$$

implying that $N_1(a_1) - N_2(a_1,-b_1;\rho) = N_2(a_1,-b_1;-\rho)$. Thus, we can use $N_1(b_1)$ and $N_2(a_1,-b_1;-\rho)$ and write the overall solution as

$$(S - de^{-rt})N_1(b_1) + (S - de^{-rt})N_2(a_1,-b_1;-\rho) - Xe^{-rT}N_2(a_2,-b_2;-\rho) - (X - d)e^{-rt}N_1(b_2).$$

Let us examine each of these four terms, using the interpretation based on the assumption of risk neutrality. The first term, $(S - de^{-rt})N_1(b_1)$, is the discounted expected value of the stock price at the ex-dividend day, conditional on the stock price exceeding the critical price. This reflects the expected receipt of the stock upon exercise of the option immediately before the ex-dividend day. $(X - d)e^{-rt}N_1(b_2)$ is the discounted expected payout at the ex-dividend day from exercise of the option. The probability term is the univariate probability of early exercise. $(S - de^{-rt})N_2(a_1,-b_1;-\rho)$ is the discounted expected value of the stock price at the expiration, given that the option was not exercised early and ends up in-the-money. The final term, $Xe^{-rT}N_2(a_2,-b_2;-\rho)$, is the discounted expected payout of the exercise price at expiration, conditional on the option not having been exercised early. Each of these interpretations is based on the assumption of risk neutrality, which as we know, is not valid but permits correct pricing of options. Thus, when we say one of these probabilities is the probability of something occurring, we mean the probability if investors were risk neutral.

Now let us observe how the formula converges to the Black-Scholes formula for the case of zero dividends. Letting $d = 0$, we have

$$SN_1(b_1) + SN_2(a_1,-b_1;\rho) - Xe^{-rT}N_2(a_2,-b_2;-\rho) - XN_1(b_2).$$

With $d = 0$, $S_t^* \rightarrow \infty$. Then $b_1 \rightarrow -\infty$ and $b_2 \rightarrow -\infty$. Then $N(b_1) = N(b_2) = 0$. So this leaves us with

$$SN_2(a_1,-b_1;-\rho) - Xe^{-rT}N_2(a_2,-b_2;-\rho).$$

The first probability in the equation above is defined as

$$N_2(a_1,-b_1;-\rho) \;=\; N_1(a_1)\int_{-\infty}^{-b_1}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\left(\frac{1}{2}\right)\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)dy.$$

With $-b_1 \rightarrow \infty$, then $N_2(a_1,-b_1;-\rho) = N_2(a_1;\infty;-\rho)$ is the joint probability that $x \leq a_1$ and $b \leq \infty$. This is simply $N_1(a_1)$. Similarly, $N_2(a_2,-b_2;-\rho) = N_2(a_2;\infty;-\rho) = N_1(a_2)$. This reduces the overall expression to

$$SN_1(a_1) - Xe^{-rT}N_1(a_2),$$

which is the Black-Scholes formula.

The Two-Dividend Case

The model has been extended to cover the case of two known dividends during the life of the option (Welch and Chen (1988) and Stephan and Whaley (1990). Let the first dividend be d_1 and the time to the ex-dividend date be t_1 and the second dividend be d_2 and the time to the second ex-dividend date be t_2. The solution is

$$\begin{aligned}
c(S,X,T,d_1,t_1,d_2,t_2) = {} & (S - d_1e^{-rt_1})(1 - N_3(-a_1,-b_1,-c_1;\sqrt{t_1/T},\sqrt{t_2/T},\sqrt{t_1/t_2})) \\
& - X[e^{-rt_1}N_1(c_2) + e - rt_2N_2(b_2,-c_2;-\sqrt{t_1/t_2}) \\
& + e^{-rT}N_3(a_2,-b_2,-c_2;-\sqrt{t_1/T},-\sqrt{t_2/T},\sqrt{t_1/t_2})) \\
& + d_1e^{-rt_1}N_1(c_2) + d_2e^{-rt_2}[N_1(c_1) + N_2(b_2,-c_2;-\sqrt{t_1/t_2})]
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\ln((S - d_1e^{-rt_1} - d_2e^{-rt_1})/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
a_2 &= a_1 - \sigma\sqrt{T} \\
b_1 &= \frac{\ln((S - d_1e^{-rt_1} + d_2e^{-rt_2})/S^*_{t_2}) + (r + \sigma^2/2)t_2}{\sigma\sqrt{t_2}} \\
b_2 &= b_1 - \sigma\sqrt{t_2} \\
c_1 &= \frac{\ln((S - d_1e^{-rt_1} + d_2e^{-rt_2})/S^*_{t_1}) + (r + \sigma^2/2)t_1}{\sigma\sqrt{t_1}} \\
c_2 &= c_1 - \sigma\sqrt{t_1}.
\end{aligned}$$

The critical stock prices are defined by the relationships,

$$\begin{aligned}
c(S^*_{t_1},X,T-t_1,d_2,t_2-t_1) &= S^*_{t_1} + d_1 - X \\
c(S^*_{t_2},X,T-t_2) &= S^*_{t_2} + d_2 - X.
\end{aligned}$$

Note that in the first case, the critical stock price at the first ex-dividend date must equate the exercise value, the right-hand side, with the value of an *American* call. This American call price would come from the one-dividend American call formula. At the second ex-dividend date, the critical price equates the exercise value with the Black-Scholes price of a European call with time $T - t_2$ remaining.

The probability $N_3(x,y,z;\rho_{xy},\rho_{xz},\rho_{yz})$ is the trivariate normal probabability. It can be calculated by defining it in relation to the univariate normal probability as follows

$$N_3(x,y,z;\rho_{xy},\rho_{xz},\rho_{yz}) = \int_{-\infty}^{y} N_1\left(\frac{y-\rho_{xy}q}{\sqrt{1-\rho_{xy}^2}}\right) N_1\left(\frac{z-\rho_{yz}q}{\sqrt{1-\rho_{yz}^2}}\right) n_1(q)dq,$$

where $n_1(q)$ is the univariate normal density function. Numerical integration can usually be used to evaluate this integral.

In the two-dividend case, the problem can sometimes be simplified. Consider the following cases, which are collectively exhaustive.

$d_1 = d_2 = 0$.	Use the Black-Scholes model
$d_1 < X(1 - e^{\{-r(t_2 - t_1)\}})$, $d_2 = 0$.	The first dividend is too small to justify early exercise. Then use the Black-Scholes model with $S - d_1e^{-rt}{}_1$ as the stock price.
$d_1 > X(1 - e^{\{-r(t_2 - t_1)\}})$, $d_2 = 0$.	Early exercise is possible at the first ex-dividend date, but not at the second ex-dividend date. Use the 1-dividend American call formula.
$d_1 < X(1 - e^{\{-r(t_2 - t_1)\}})$ and $d_2 < X(1 - e^{\{-r(T - t_2)\}})$.	Both dividends are too small to justify early exercise. Then use $S - d_1e^{-rt}{}_1 - d_2e^{-rt}{}_2$ as the stock price in the Black-Scholes model.
$d_1 < X(1 - e^{\{-r(t_2 - t_1)\}})$ and $d_2 > X(1 - e^{\{-r(T - t_2)\}})$.	The option will not be exercised at the first ex-dividend date because d_1 is too small, but it could be exercised at the second ex-dividend date. Use the 1-dividend American call formula,

but you will still need to subtract the present value of both dividends from the stock price.

$d_1 > X(1 - e^{\{-r(t_2 - t_1)\}})$ and
$d_2 < X(1 - e^{\{-r(T - t_2)\}})$. The option could be exercised at the first ex-dividend date but not the second. Use the 1-dividend American call formula, but you will still need to subtract the present value of both dividends from the stock price.

$d_1 > X(1 - e^{\{-r(t_2 - t_1)\}})$ and
$d_2 > X(1 - e^{\{-r(T - t_2)\}})$. The option could be exercised at either ex-dividend date. Use the 2-dividend American call formula.

For more than two dividends, the model would extend in the same manner. For n dividends, one would be required to evaluate an (n+1)-variate normal probability distribution, but computation of multivariate normal integrals is quite difficult. In that case one usually would use a numerical method such as a binomial or finite difference approach.

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