TEACHING NOTE 98-03:

CLOSED-FORM AMERICAN PUT OPTION PRICING:

GESKE-JOHNSON

It is well-known that the price of an American put follows the partial differential equation

$$\frac{\partial P}{\partial t} = rP - rS\frac{\partial P}{\partial S} - \frac{1}{2}\frac{\partial^2 P}{\partial S^2}\sigma^2 S^2.$$

For a European put (using lower case p in the above equation), the boundary condition is simply

$$p(S_T,T) = Max(0, X - S_T)$$

where S_T is the asset price at expiration. For an American put we require the additional condition,

$$P(S_t,T) \geq Max(0, X - S_t) \ \forall \ t < T.$$

This condition simply means that at any time t prior to expiration, the American put value must be at least its exercise value since it can be exercised immediately. Since t and S change, this is a free boundary problem and had for many years been thought to be unsolvable. Geske and Johnson (1984), however, obtained a solution as an n-fold compound option, using Geske's (1979) compound option formula and the equivalent martingale/risk neutrality assumption.

Consider the layout of continuous time: From t to t+dt is time unit dt. From t to time t + 2dt is time unit 2dt, and so on. At any instant the put can be exercised. If it is not exercised at that instant, we move forward and consider whether it is optimal to exercise the put at the next instant. The exercise decision will be determined by many factors and will take into account that optimality may come from waiting until later to exercise it.

To solve the problem we need to know the correlation between the Wiener processes, dz(1) and dz(1,2) defined where dz(1,2) = dz(1) + dz(2) and dz(1) and dz(2) are the stochastic shocks at times 1 and 2, and more generally any two consecutive time points. Note that dz(1,2) is the cumulative disturbance over the two time increments. We can obtain the necessary correlation with the following results:

$$\begin{aligned}
\rho_{12} &= \frac{cov(dz(1),dz(1,2))}{\sqrt{var(dz(1))var(dz(1,2))}} \\
&= \frac{cov(dz(1),dz(1)+dz(2))}{\sqrt{var(dz(1))var(dz(1,2))}} \\
&= \frac{cov(dz(1),dz(1))+cov(dz(1),dz(2))}{\sqrt{var(dz(1))var(dz(1,2))}} \\
&= \frac{var(dz(1))+0}{\sqrt{var(dz(1))var(dz(1,2))}},
\end{aligned}$$

which uses the fact that $dz(1)$ and $dz(2)$ are independent. Now, $var(dz(1,2)) = var(dz(1)) + var(dz(2)) + 2cov(dz(1),dz(2))$, which equals $dt + dt + 0 = 2dt$. Consequently,

$$\rho_{12} = \sqrt{\frac{dt}{2dt}}.$$

This is usually expressed as

$$\rho_{12} = \sqrt{t_1/t_2}$$

where t_1 is the time elapsed over one increment and t_2 is the time elapsed over two increments.[1]

At any instant we exercise the option if

(a)   it has not already been exercised, and

(b)   if the payoff from exercising is greater than the value of the put if it is not exercised.

The critical asset price that triggers early exercise is defined as S_t^* in the equation,

$$X - S_t^* = P(S_t^*,t).$$

At the first instant there is no probability of prior exercise. So we integrate the exercise payoff, which is the exercise price minus the stock price, over all possible stock prices less than the critical stock price. Then we discount this value back one instant. This produces two terms, one of which is the discounted exercise price times the probability that the stock price will be below the critical stock price. At the next instant we must also discount the expected payoff, but we must consider the joint

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[1]The derivation of ρ_{12} as this expression is not provided in the Geske-Johnson paper. It is added here for clarification.

probability of exercise at that instant conditional on not having exercised at the previous instant. We follow this procedure for the remaining instants over the life of the option.

Ultimately the price of the option then becomes a discounted weighted average of all of the possible payoffs. It can be written as follows:

$$e^{-rdt}(X - S_{dt}^*)Prob(S_{dt} \leq S_{dt}^*$$
$$+ e^{-r2dt}(X - S_{2dt}^*)Prob(S_{2dt} \leq S_{2dt}^* \text{ and } S_{dt} > S_{dt}^*)$$
$$+ e^{-r3dt}(X - S_{3dt}^*)Prob(S_{3dt} \leq S_{3dt}^*, S_{2dt} > S_{2dt}^* \text{ and } S_{dt} > S_{dt}^*$$
$$+ \ldots$$

continuing until T.

The overall solution can be written more compactly as

$$P = Xw_2 - Sw_1$$

where

$$w_1 = \{N_1(-d_1(S_{dt}^*,dt) + N_2(d_1(S_{dt}^*,dt),-d_1(S_{2dt}^*,2dt);-\rho_{12})$$
$$+ N_3(d_1(S_{dt}^*,dt),d_1(S_{2dt}^*,2dt),-d_1(S_{3dt}^*,3dt);\rho_{12},-\rho_{13},-\rho_{23}) + \ldots\}$$
$$w_2 = \{e^{-rdt}N_1(-d_2(S_{dt}^*,dt) + e^{-r2dt}N_2(d_2(S_{dt}^*,dt),-d_2(S_{2dt}^*,2dt);-\rho_{12})$$
$$+ e^{-r3dt}N_3(d_2(S_{dt}^*,dt),d_2(S_{2dt}^*,2dt),-d_2(S_{3dt}^*,3dt);\rho_{12},-\rho_{13},-\rho_{23}) + \ldots\}$$

and

$$d_1(q,\tau) = \frac{\ln(S/q) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$
$$d_2(q,\tau) = d_1 - \sigma\sqrt{\tau}$$

for any q and τ. The correlation coefficients are defined as

$$\rho_{12} = \sqrt{\frac{dt}{2dt}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\rho_{13} = \sqrt{\frac{dt}{3dt}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\rho_{23} = \sqrt{\frac{2dt}{3dt}} = \sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}}.$$

Geske and Johnson provide the partial derivatives with respect to each of the terms that go into the formula and Blomeyer and Johnson (1988) provide an extension that takes into account dividend payments and also conduct empirical tests of the model.

Since exercise is possible at any time instant, this formula consists of an infinite number of terms, which would seem to make the formula analytically untractable. Geske and Johnson emphasize that the formula is an exact solution, even if it does contain an infinite number of terms. They provide an analytical approximation based on a method called Richardson extrapolation, which basically assumes that the put can be exercised only at three points spread over its life. This approach greatly reduces the complexity of the problem. They obtain the price of a put that can be exercised only at expiration, the standard Black-Scholes model price, plus the price of another put that can be exercised only halfway to expiration or at expiration, which requires a bivariate normal probability computation, plus the price of another put that can be exercised only one-third of the way through its life or two-thirds of the way through its life or at expiration, which requires a trivariate normal probability computation. The American put value is then a weighted average of the prices of these three puts. They argue that an approximation of an exact formula is better than an approximation where the solution is not known, which, other than using numerical methods, is all one can do to price an American put. Bunch and Johnson (1992) provide an improvement on this technique.

References

Blomeyer, E. C. and H. Johnson. "An Empirical Examination of the Pricing of American Put Options." *Journal of Financial and Quantitative Analysis* 23 (1988), 13-22.

Bunch, D. S. and H. Johnson. "A Simple and Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach." *The Journal of Finance* 47 (1992), 809-816.

Geske, R. and H. E. Johnson. "The American Put Option Valued Analytically." *The Journal of Finance* 39 (1984), 1511-1524.