TEACHING NOTE 00-03:

MODELING ASSET PRICES AS STOCHASTIC PROCESSES II

This note continues TN96-04, Modeling Asset Prices as Stochastic Processes I. It derives the stochastic process for the asset price in a heuristic manner. We obtained

$$\frac{dS_t}{S_t} = \alpha\,dt + \sigma\,dW_t.$$

The variable dW_t is the increment to a Brownian Motion. Recall that dW_t is normally distributed with $E(dW_t) = 0$, $Var(dW_t) = dt$, and dW_t^2 is non-stochastic and equal to dt. From these results we state the following:

$$E\left(\frac{dS_t}{S_t}\right) = \alpha\,dt$$

$$Var\left(\frac{dS_t}{S_t}\right) = \sigma^2\,dt.$$

Given the fact that dS_t/S_t is just a linear transformation of a normally distributed random variable dW_t, then it is also normally distributed. In this note, we formally derive this stochastic process and some important results related to it.

The relative return on the asset over the period of time 0 to time dt is

$$\frac{S_{dt}}{S_0}.$$

The return from time dt to time 2dt is

$$\frac{S_{2dt}}{S_{dt}}.$$

This pattern continues so that at a given future time T, the return is

$$\frac{S_T}{S_{T-1}}.$$

The overall return on the asset from time 0 to time T is

$$\frac{S_T}{S_0}.$$

This return can be expressed as

$$\frac{S_T}{S_0} = \left(\frac{S_{dt}}{S_0}\right)\left(\frac{S_{2dt}}{S_{dt}}\right) \cdot \cdot \cdot \left(\frac{S_{T-1}}{S_{T-2}}\right)\left(\frac{S_T}{S_{T-1}}\right).$$

Suppose we convert the return above into the log return.

$$\ln\left(\frac{S_T}{S_0}\right) = \ln\left(\frac{S_{dt}}{S_0}\right) + \ln\left(\frac{S_{2dt}}{S_{dt}}\right) \cdot \cdot \cdot + \ln\left(\frac{S_{T-1}}{S_{T-2}}\right) + \ln\left(\frac{S_T}{S_{T-1}}\right).$$

We see that the log return for the period of time 0 to time T is the sum of the log returns of the sub-periods during time 0 to time T. The central limit theorem says that a random variable that is the sum of other random variables approaches a normal distribution. Thus, we know that the return from time 0 to time T is normally distributed. In turn we can propose that any of the sub-periods is infinitesimally small such that it, too, is made up of a series of component returns over infinitesimally smaller sub-periods. Hence, we propose that the return over any arbitrary period from t to t + dt is normally distributed with expectation of μ and variance of σ^2.

$$\ln\left(\frac{S_{t+dt}}{S_t}\right) \sim N(\mu, \sigma^2).$$

We can specify the log return in the following manner:

$$\ln\left(\frac{S_{t+dt}}{S_t}\right) = \ln S_{t+dt} - \ln S_t = d\ln S_t.$$

We then propose that the log return follows the stochastic process

$$d\ln S_t = \mu\,dt + \sigma\,dW_t,$$

where the expectation and variance are, therefore,

$$E(d\ln S_t) = \mu\,dt$$
$$Var(d\ln S_t) = \sigma^2\,dt.$$

From here we want the return dS_t. Let us propose the following transformation:

$$G_t = \ln S_t,$$

so that

$$S_t = \exp(G_t).$$

Now, temporarily dropping the time subscript, we apply Itô's Lemma to S_t:

$$dS = \frac{\partial S}{\partial G}dG + \frac{1}{2}\frac{\partial^2 S}{\partial G^2}dG^2.$$

The partial derivatives are easily obtained as

$$\frac{\partial S}{\partial G} = \exp(G) = S \qquad \frac{\partial^2 S}{\partial G^2} = \exp(G) = S.$$

Substituting these results, we get

$$dS = SdG + \frac{1}{2}SdG^2.$$

Since $dG = d\ln S$, the differentials, dG, and dG^2, are

$$dG = (u\,dt + \sigma\,dW) \qquad dG^2 = \sigma^2\,dt,$$

with the second result making use of the fact that any power of dt greater than one is zero. Substituting these results, we obtain

$$dS = S(u\,dt + \sigma\,dW) + \frac{1}{2}S\sigma^2\,dt.$$

Dividing both sides by S_t and adding the time subscript, we now have the stochastic process for dS_t,

$$\frac{dS_t}{S_t} = (\mu + \sigma^2/2)dt + \sigma\,dW_t.$$

Defining $\alpha = \mu + \sigma^2/2$, we have

$$\frac{dS_t}{S_t} = \alpha\,dt + \sigma\,dW_t.$$

The expectation and volatility are

$$E\left(\frac{dS_t}{S_t}\right) = \alpha\,dt$$

$$Var\left(\frac{dS_t}{S_t}\right) = \sigma^2\,dt.$$

Thus, we now have the stochastic differential equations for the return and the log return. The return over the longer period is S_T/S_0. The log of this, i.e., the log return over the longer period, is normally distributed. That means that S_T/S_0 is lognormally distributed. Both the infinitesimal return, dS_t/S_t, and the infinitesimal log return, $d\ln S_t$, are normally distributed.

Solving the Stochastic Differential Equation

The equations for the return and log return are stochastic processes, as well as stochastic differential equations. A differential equation has a potential solution, which is a function such that the derivatives conform to the differential equation. In this context, a solution would be the stock price at some time t, expressed in terms of the stock price at a previous time such as time 0.

To obtain S_t in terms of S_0, we take the equation for the log return and set up to integrate over the time interval 0 to t:

$$\int_0^t dG_j = \int_0^t \mu \, dj + \int_0^t \sigma \, dW_j.$$

The left-hand side is clearly $G_t - G_0$. The first integral on the right-hand side is a standard Riemann integral and becomes

$$\int_0^t \mu \, dj = \mu \int_0^t dj = \mu t.$$

The second integral on the right-hand side is a stochastic integral and one of the simplest of all stochastic integrals. It is obtained as

$$\int_0^t \sigma \, dW_j = \sigma \int_0^t dW_j = \sigma(W_t - W_0).$$

In fact, in this case, the stochastic integral is so simple, it is the same as the Riemann integral. The value W_t is the value of the Brownian motion process at time t. It is quite common that W_0 is set at zero. So we have σW_t. Then $G_t - G_0 = \mu t + \sigma W_t$. Since $S_t = \exp(G_t)$, and thus, $S_0 = \exp(G_0)$,

$$S_t = S_0 e^{\mu t + \sigma W_t}.$$

We can check to see if this is the solution by using Itô's Lemma on S_t:

$$dS_t = \frac{\partial S_t}{\partial W_t} dW_t + \frac{\partial S_t}{\partial t} dt + \frac{1}{2} \frac{\partial^2 S_t}{\partial W_t^2} dW_t^2.$$

We obtain the partials by differentiating the solution: $\partial S_t / \partial W_t = S_t \sigma$, $\partial^2 S_t / \partial W_t^2 = S_t \sigma^2$ and $\partial S_t / \partial t = S_t \mu$. Now, recall that $dW_t^2 = dt$. Substituting all of these results and rearranging, we obtain:

$$dS_t / S_t = \alpha \, dt + \sigma \, dW_t.$$

This is the original stochastic process. Thus, our solution is correct.

Why Solutions to Stochastic Differential Equations are Not Always the Same as Solutions to Ordinary Differential Equations

Let us see how solving a stochastic differential equation is different from solving an ordinary differential equation. Consider the ordinary differential equation (ODE):

$$dY_t = Y_t dW_t,$$

where W_t is non-stochastic. This is a fairly simple ODE. We start by expressing it as

$$\frac{1}{Y_t} \frac{dY_t}{dW_t} dW_t = dW_t.$$

We now perform integration over 0 to t:

$$\int_0^t \frac{1}{Y_s} \frac{dY_s}{dW_s} dW_s = \int_0^t dW_s.$$

With $W_0 = 0$, the solution is $\ln Y_t = W_t$ or $Y_t = \exp(W_t)$.

Now we let W_t be stochastic. We start by proposing a general form for the solution. Specifically, we shall say that $Y_t = \exp(X_t)$. In other words, X_t is some function that solves the equation and in which X_t is a function of W_t. In the special case $X_t = 0$, giving $Y_0 = 1$. In the ODE case, $X_t = W_t$. First we use Itô's Lemma on X_t and obtain:

$$dX_t = \frac{\partial X_t}{\partial Y_t} dY_t + \frac{1}{2} \frac{\partial^2 X_t}{\partial Y_t^2} dY_t^2.$$

The partial derivatives are $\partial X_t/\partial Y_y = 1/Y_t$ and $\partial^2 X_t/\partial Y_t^2 = -(1/Y_t^2)$. We also have that $dY_t = Y_t dW_t$ and $dY_t^2 = Y_t^2 dt$, due to the properties of dW_t. Substituting these results, we obtain

$$dX_t = dW_t - \frac{1}{2} dt.$$

Now we perform the integration,

$$\int_0^t dX_s = \int_0^t dW_s - \int_0^t \frac{1}{2} ds,$$
$$X_t - X_0 = W_t - t/2$$

With $X_t = \ln Y_t$, then

$$Y_t = e^{W_t - t/2}.$$

Notice that now we have an additional term $t/2$. Thus, at least in this common situation, and quite often otherwise, the solution to an SDE is not the same as a solution to an ODE.

Finding the Expected Future Stock Price

Given the solution,

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

to the stochastic differential equation, we shall now use it to obtain the expected stock price at t. Using the above we express the problem as follows:

$$E[S_t] = S_0 E[e^{\mu t + \sigma W_t}]$$
$$= S_0 e^{\mu t} E[e^{\sigma W_t}].$$

This expectation is easily evaluated by recognizing that W_t is normally distributed. We are reminded that the probability density for a normally distributed random variable W_t, which has mean zero and variance t is

$$f(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-W_t^2/2t}.$$

Thus, we can find the expected value of S_t by evaluating the following expression:

$$E[e^{\sigma W_t}] = \int_{-\infty}^{\infty} e^{\sigma W_t} \frac{1}{\sqrt{2\pi t}} e^{-W_t^2/2t} dW_t.$$

Write the right-hand side as

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{\sigma W_t} e^{-W_t^2/2t} dW_t.$$

Work on the exponent

$$\sigma W_t - W_t^2/2t = \frac{2\sigma W_t t - W_t^2}{2t}$$

$$= \frac{2\sigma W_t t - W_t^2}{2t} + \frac{\sigma^2 t}{2} - \frac{\sigma^2 t}{2}$$

$$= \frac{2\sigma W_t t - W_t^2 - \sigma^2 t^2}{2t} + \sigma^2 t/2$$

$$= -\frac{1}{2}\frac{(W_t - \sigma t)^2}{t} + \sigma^2 t/2.$$

So now we have

$$e^{\sigma^2 t/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\left(\frac{W_t - \sigma t}{\sqrt{t}}\right)^2} dW_t.$$

The integrand is the probability density function for a normally distributed random variable with mean σt and variance t and, by definition, integrates to a value of 1.0. Thus,

$$E[e^{\sigma W_t}] = e^{\sigma^2 t/2}.$$

So our expectation is,

$$E[S_t] = S_0 e^{\mu t + \sigma^2 t/2}$$

$$= S_0 e^{(\mu + \sigma^2/2)t}.$$

Note that this result is also equal to $E[S_t] = S_0 e^{\alpha t}$. This is an intuitively simple result. It says that the expected future stock price is the current stock price compounded at the expected rate of return.

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