On Various Kinds of Rounding Rules for Multiplication and Division

Wei-Da Chen,[1] Wei Lee,[1,*] and Christopher L. Mulliss[2]

[1]*Department of Physics, Chung Yuan Christian University, Chung-Li, Taiwan 32023, R.O.C.*
[2]*Ball Aerospace and Technologies Corporation, Fairborn, Ohio 45324, U.S.A.*

(Received January 8, 2004)

A detailed investigation of three different rounding rules for multiplication and division is presented, including statistical analyses via Monte-Carlo simulations as well as a mathematical derivation. This work expands upon a previous study by Mulliss and Lee (1998), by making the more realistic assumption that the contributing uncertainties are statistically independent. With this assumption, it is shown that the so-called standard rounding rule fails over 60% of the time, leading to a loss in precision. Two alternative rules are studied, and both are found to be significantly more accurate than the standard rule. One alternative rule requires one extra significant digit beyond that predicted by the standard rule. The other requires one to count numbers whose leading digit is 5 or greater as having an extra significant digit, and then to apply the standard rule. Although the second alternative rule is slightly more accurate, the first is shown to be completely safe for data — never leading to a truncation of digits that contain significant information. Accordingly, we recommend the first alternative rule as the new standard.

PACS numbers: 01.30.Pp, 01.55.+b, 02.70.Uu

I. INTRODUCTION

Over the years, college physics students and teachers have come to rely on a standard rounding rule, which states that the proper number of significant figures in the result of multiplication or division is the same as the smallest number of significant figures in any of the numbers used in the calculation. However, Good, in 1996, used a simple division problem to sound an alarm that the application of this standard rounding rule could bring about a serious loss of precision in the result [1]. What is even more disturbing is the long list of popular physics textbooks at the first-year university level which advocate the standard rounding rule, without warning their readers that data could be jeopardized due to the possibility of losing valuable information. The fact that rounding rules can give rise to such a loss is attributed to their approximate nature and has been well documented by Schwartz [2]. Some researchers feel that the approximate nature of significant figures and rounding rules precludes the need for a detailed investigation of their effects on error propagation [3]. Others, including the authors, however, recognize the importance of rounding rules as common and convenient (even if approximate) tools in error analysis [1, 4], especially for students of introductory physics and chemistry. The theoretical basis and a statistical analysis of the standard rounding rule were presented in our previous publication [5], in which we also proposed an equally simple but more accurate alternative rule (ML rule for short), requiring one extra significant figure than predicted by the standard rounding rule.

Recently, our team has been informed that there is another rounding rule for multiplication/division that has been taught in some Portuguese schools since approximately 1930 [6]. Unlike the ML rule, this rule (CLM rule for short) requires one to count numbers whose leading digit is 5 or greater as having an extra significant digit and then to apply the standard rule. In this work, we investigate the CLM rule and provide a statistical analysis as well as the mathematical basis for this rule. As was categorized in our previous work [5], the statistical results fall into three cases: those where the true uncertainty is as large or larger than that predicted by the rule but of the same order of magnitude, those where the true uncertainty is less than predicted and those where the true uncertainty is an order of magnitude larger than predicted. The rule is said to "work" in the first case because it predicts the minimum number of significant digits that can be written down without losing precision and hence valuable information in the result. In the second and third cases, the rule apparently "fails", predicting fewer or more significant figures than are needed and, accordingly, losing or overstating the precision.

To properly compare these three rounding rules, similar statistical tests are reproduced for the standard rule and the ML rule. This study indicates that the standard rounding rule is inferior to both the alternative rules, of which the CLM rule has the highest accuracy (66%). The ML rule, although slightly less accurate than the other alternate rule, never leads to the truncation of digits that contain useful information. This assures the ML rule a place in calculations where the complete safety for data is strictly required.

II. SIMPLE MATHEMATICAL DERIVATION FOR ROUNDING RULES

The rounding rules for the multiplication/division of two numbers can be inferred from one very simple assumption, which states that the precision (percentage error) of a number is approximately related to the number of significant figures in that number [7]. The fundamental principles that lead to this derivation were discussed in earlier literature, but not explicitly developed in a rigorous mathematical manner [8]. Written in mathematical form, this assumption expresses the precision in a number x with N_x significant figures in the form:

$$\text{Precision}(x) \simeq 10^{(2-N_x)}\% \,. \tag{1}$$

Following Bevington and Robinson [9], the absolute error in this number is taken to $\pm 1/2$ in the least significant decimal place. However, Eq. (1) is only an approximate relationship. In reality, it is modified to [5]

$$\text{Precision}(x) = C_x \times 10^{(2-N_x)}\% \,, \tag{2}$$

where C_x is a constant that can range from approximately 0.5 to exactly 5, depending on the actual value of the number x. Table I displays the relationship.

To obtain a formula that determines the number of significant figures in a product or ratio, consider the simple division problem $x = y/z$. Through differentiation, it is

TABLE I: Examples illustrating Eq. (2).

Number x	Precision(%)	Number of significant figures N_x	Value of C_x
6	0.83×10^{1}	1	0.83
72	0.69×10^{0}	2	0.69
64.4	0.78×10^{-1}	3	0.78
92.37	0.54×10^{-2}	4	0.54

customary to use the following relation in the simplest approximation [5]

$$\max(dx/x) = \mathrm{abs}(dy/y) + \mathrm{abs}(dz/z)\,, \tag{3}$$

or

$$\mathrm{Precision}(x) = \mathrm{Precision}(y) + \mathrm{Precision}(z)\,. \tag{4}$$

The relation given in Eq. (4) is strictly valid only when the uncertainties in y and z are perfectly correlated. For most real-world problems, this occurs very seldom and the above relation places an upper limit on the uncertainty in the result x. In the previous investigation of rounding rules for multiplication and division, the authors used the above relation [5]. A more appropriate assumption for most real-world problems is that the uncertainties in y and z are statistically independent. This more realistic assumption takes into account the concepts of variance/covariance and leads to the following error-propagation equation [9]:

$$(\mathrm{Precision}(x))^2 = (\mathrm{Precision}(y))^2 + (\mathrm{Precision}(z))^2\,. \tag{5}$$

Assuming that $N = \min(N_y, N_z)$ and $N' = \max(N_y, N_z)$, the substitution of Eq. (2) into Eq. (5) yields the following expression of N_x:

$$N_x = N + \left[\log(C_x) - \frac{1}{2}\log\left[C^2 + C'^2 \times 10^{-2(N'-N)}\right]\right]\,, \tag{6}$$

where C and C' correspond to N and N', respectively. This is the formula allowing one to determine the number of significant figures of a product or ratio.

III. A STATISTICAL STUDY OF THE THREE DIFFERENT ROUNDING RULES

III-1. The method

To investigate the statistical properties of the three rounding rules, a Monte-Carlo procedure was used. A computer code was written in Fortran 90 and run on a Pentium(R) 4

TABLE II: The statistical results of the application of the standard rounding rule to simple multiplication and division problems. The statistical likelihood that the application of the standard rounding rule will fall into each of the three categories described in the text is shown.

Category	Multiplication	Division	Average
1 more digit needed	68.8%	54.4%	61.6%
Worked	31.0%	45.3%	38.2%
1 digit too many	0.2%	0.3%	0.2%

PC equipped with a Microsoft Fortran PowerStation compiler (Version 4.0). The code uses a random number generator based upon Ran2 [10] to create two numbers. Each of these numbers has a randomly determined number of significant figures ranging from 1 to 5 and a randomly determined number of places to the left of the decimal point ranging from 0 to 5. Each digit in these numbers is randomly assigned a value from 0 to 9, except for the leading digit which is randomly assigned a value from 1 to 9. Each resulting number can range from the smallest and least precise value of 0.1 to the largest and most precise value, 99999. The program calculates the product or ratio of the two generated numbers and determines the number of significant figures which should be kept according to each of the three rules. It then uses Eqs. (2) and (5) to compute the "true" precision of the product/ratio and converts it into a true absolute error. The absolute error in the product/ratio predicted by each rule is taken to $\pm 1/2$ in the least significant decimal place. The true absolute error and the value predicted by each rule are compared, and the multiplication or division problem is counted as one of the three cases described previously. The program repeats the calculation for one million multiplication or division problems and computes statistics.

III-2. The standard rounding rule

Table II shows the results for the standard rounding rule as applied to simple multiplication and division problems. As can be noted, the result differs from that in our previous work [5]. One reason might be that the simulations were run on different computer hardware systems using different software codes. The primary reason is due to the fact that our current statistical analysis assumes that uncertainties are statistically independent and uses Eq. (5), while the previous work assumed that uncertainties are perfectly correlated and used Eq. (4). We see that the current statistical analysis estimates a lower accuracy for the standard rule than was estimated in our previous work. This is exactly what is intuitively expected since the assumption of statically independent uncertainties generally results in a smaller predicted uncertainty in the result. This smaller uncertainty, in turn, makes it more likely that the standard rule will predict too few significant figures in the result.

To see the mathematical background of the standard rounding rule, let us substitute the approximate relationship Eq. (1) into Eq. (5); we then have

$$10^{-2N_x} = 10^{-2N_y} + 10^{-2N_z} \,. \tag{7}$$

If $N_y \neq N_z$, and we assume $N_y < N_z$, then the 10^{-2N_y} term is at least 100 times larger than

TABLE III: The statistical results of the application of the ML rule to simple multiplication and division problems.

Category	Multiplication	Division	Average
More digits needed	0	0	0
Worked	68.8%	54.7%	61.8%
Too many digits	31.2%	45.3%	38.2%

the 10^{-2N_z} term and thus completely dominates. We find that $N_x = N_y = \min(N_y,\ N_z)$, which is a statement of the standard rounding rule. As for the case in which $N_y = N_z$, we can conclude that $N_x = \mathrm{NINT}[N_y - \log(\sqrt{2})] = N_y = \min(N_y, N_z)$, where $\mathrm{NINT}(w)$ denotes the integer that is nearest to the number w. In both cases (and in general, therefore) we obtain a statement of the standard rounding rule. Obviously, the reason why the standard rounding rule fails is that it is an approximation, in which we neglect the contribution of the leading constant C_x to the precision.

III-3. The ML rule

Complex as Eq. (6) may be, some useful properties can still be extracted. By studying the bracketed term in Eq. (6), one can show that the number of significant figures predicted by the standard rounding rule can never be more than one digit away from the correct value [5]. The only possible values for N_x are $N-1$, N, or $N+1$. Thus, the standard rounding rule is never "wrong" by more than one significant digit. Based upon this reason the ML rule, which requires one to use an *extra* significant digit above that suggested by the standard rule, has been proposed. The statistical results are shown in Table III. Again, the result differs slightly from that in our previous work, owing to the reasons mentioned already.

It should be pointed out that the most significant aspect of the ML rule is that it never leads to a loss of precision. The reason for this arises from the fact that the standard rounding rule can, at its worst, predict only one less significant digit than actually needed. The "extra" significant digit that the ML rule provides comes to the rescue. The ML rule is not only more accurate than the standard rounding rule, it is also completely safe for data. The only drawback with the ML rule is that the results of calculations may have one or, in rare cases, two too many significant digits. This disadvantage is minor when compared with the standard rounding rule.

III-4. The CLM rule

Apart from the two rounding rules mentioned above, there is also a third rounding rule (the CLM rule) that has been taught in some Portuguese schools at least since 1930 [6]. This rule requires one to count numbers whose leading digit is 5 or greater as having an extra significant significant digit and then to apply the standard rule. Although this rule can be stated very simply, it is more complicated than the other two rules, because there are the following two distinct cases that must be considered:

(a) When the numbers have the same number of significant digits: add an extra digit to

TABLE IV: Some particular values of C_x.

a_x	C_x	a_x	C_x
1	5	6	0.83
2	2.5	7	0.71
3	1.66	8	0.63
4	1.25	9	0.56
5	1	< 10	> 0.5

the answer if both numbers have a leading digit of 5 or greater. (b) When the numbers do not have the same number of significant digits: add one extra digit to the answer if the number with the fewer significant figures has a leading digit of 5 or greater.

The CLM rule deals with the value of the leading digit. In order to understand the mathematical basis of the CLM rule, one must relate the number of significant digits in the result to the value of the leading digits. Let us express a number x in scientific notation as $x = a_x \times 10^n$, where $1.0 \leq a_x < 10.0$ and n is an integer. Then it can easily be shown that the leading constant C_x in precision is related to a_x by the following equation:

$$C_x = 5/a_x\,. \tag{8}$$

One can clearly see that C_x ranges from 0.5 to exactly 5, depending on the actual value of the number x (i.e., the actual numerical value of a_x). To the first order, a_x can be replaced by the value of the leading digit for the purposes of computing C_x. Table IV gives some particular values of a_x and the corresponding values of C_x.

An understanding of the mathematical basis of the CLM rule can now be obtained from a close examination of Eq. (6). Assuming that $N' > N$ and ignoring the small contribution from $C'^2 \times 10^{-2(N'-N)}$, the substitution of Eq. (8) into Eq. (6) results in the following approximate relationship:

$$N_x = N + \log\left(a/a_x\right), \quad (N' > N)\,. \tag{9}$$

Assuming that $N' = N$ results in the following relationship:

$$N_x = N + \log\left(a/a_x\right) - (1/2)\log\left[1.0 + (a/a')^2\right], \quad (N' = N)\,. \tag{10}$$

In Eqs. (9) and (10), a is (to first order) the leading digit of the number with fewer significant figures, a_x is (to first order) the leading digit of the result, and a' is (to first order) the leading digit of the number with more significant figures.

In the case where $N' > N$, Eq. (9) clearly shows that the proper number of significant figures in the result, N_x, is more likely to be $N + 1$ than N, when the leading digit of the number with fewer significant figures, a, is large. Considering the possible values of a_x, the smallest value of a that can cause Eq. (9) to be rounded up to $N_x = N + 1$ is $a = 4$. The value of $a = 5$ is the smallest value at which it is equally likely (averaged over the possible values of a_x) that the $\log\left(a/a_x\right)$ term will cause N_x to be greater than N. Values of a

TABLE V: Contribution of the leading digit a to the number of significant figures in the result, N_x.

Value of a	Possible values of $\log(a/a_x)$		
(to first order)	Minimum	Average	Maximum
1	−0.954	−0.699	+0.000
2	−0.653	−0.398	+0.301
3	−0.477	−0.222	+0.477
4	−0.352	−0.097	+0.602
5	−0.255	+0.000	+0.699
6	−0.176	+0.080	+0.778
7	−0.109	+0.146	+0.845
8	−0.051	+0.204	+0.903
9	+0.000	+0.255	+0.954

TABLE VI: The statistical results of the application of the CLM rule to simple multiplication and division problems.

Category	Multiplication	Division	Average
1 more digit needed	26.97%	16.95%	22.0%
Worked	62.02%	68.99%	65.5%
1digit too many	11.01%	14.06%	12.5%

greater than 5 begin to favor $N_x = N + 1$ over $N_x = N$ more strongly; see Table V. There is obviously a strong element of truth to the CLM rule for the case where $N' > N$, and it can be anticipated that the CLM rule will be more accurate than the standard rounding rule for this case.

The $N'{=}N$ case is complicated by an additional negative term, i.e., $-(1/2)\log[1.0 + (a/a')^2]$, that appears in Eq. (10). This term makes it less likely, in general, that an additional significant figure will be needed in the result for the $N' = N$ case than for the $N' > N$ case. This additional term is always negative, and it becomes more negative with increasing values of a. The rate of increase of the $\log(a/a_x)$ term with increasing values of a (averaged over all possible values of a_x) is faster than the rate of decrease of the $-(1/2)\log[1.0 + (a/a')^2]$ term with increasing values of a (averaged over all possible values of a'). Thus, there is still an overall trend that makes the number of significant figures in the result, N_x, more likely to be $N + 1$ than N when the leading digit of a is large, although this trend is not as strong as it is for the $N' > N$ case. We can see from Eq. (10) that the effects of this additional negative term are minimized when the value of a' is large (e.g. 5 or greater). An investigation of Eq. (10) found that both a and a' were required to be 5 or greater before the range of possible values for N_x favored values greater than N. The above analysis shows that there is also an element of truth to the CLM rule for the case where $N' = N$.

Table VI shows the statistical results. This third rule works better for division than the other two rounding rules mentioned above. Although the CLM rule is the most accurate of the three rules examined (it is only slightly more accurate than the ML rule), its use has a significant chance of causing damage to data. As anticipated from the mathematical analysis of the CLM rule, the accuracy for the $N' > N$ case (which was computed separately but not shown separately in Table VI) was significantly better than the accuracy for the $N' = N$ case.

The detailed mathematical investigation of the CLM rule, presented here for the first time to our knowledge, led to an expression for the number of significant figures in the result, N_x, in terms of the values of the leading digits. This reformulation, in turn, provides the key for explaining why multiplication and division have different levels of accuracy for the CLM rounding rule. This is due to the fact that the multiplication and division of random numbers create different frequency distributions of a_x, the leading digit in the result. When a constraint (e.g. $a \geq 5$ or $a < 5$) is imposed on the numbers used in the calculation, there is a change (different for multiplication and division) in the frequency distribution of a_x. In short, the leading digits a and a_x in Eqs. (9) and (10) are not independent of each other. There is a statistical relationship between them that is different for multiplication and division, and that depends upon the constraints imposed on a. Readers who are interested in a detailed discussion explaining the above-mentioned general behavior can refer to the Appendix.

IV. SUMMARY

The best expression for the result of a calculation should include a precise description of the uncertainty in terms of the absolute or percentage error. This is, in reality, often only possible for experimental data. In most situations, the problems encountered by physics students in daily life, including those in textbooks, do not deal with quantities where the uncertainties are explicitly stated. When this is the case, the number of significant figures is the only available information upon which to base an error estimate and a rounding rule becomes more meaningful.

It is not convenient to use Eq. (5) to calculate the precision of a product or ratio whenever one is doing a multiplication or division problem; as a result there arises the need for a simple and quick way to determine the number of significant figures in simple mathematical operations. This is why a rounding rule is desired. The standard rounding rule is conservative, because it tries to ensure that the true result of a calculation is included within the error bars implied by the number of significant digits to which that result is written. However it is too conservative, so that it causes a loss of precision over 60% of the time. Consequently, an equally simple but more accurate rounding rule has been called for [1]. There being no *a priori* rule two alternative rules are proposed, and both are shown to be far superior to the standard rule. The first alternate rule, the ML rule, is shown to work 62% of the time while the second alterative rule, the CLM rule, works 66% of the time. It is worth noting that the ML rule, although less accurate than the CLM rule, never leads to a loss in the information carried by the digits. Concentrating on accuracy alone, the

CLM rule is recommended strongly over the standard rule and slightly over the ML rule. Considering the preservation of information alone, the ML rule is clearly preferred. Seeing that the ML rule is not only accurate but also completely safe for data, we suggest adopting the ML rule as the new standard for general purposes. One should, however, decide which rule to use depending on one's own purpose and situation.

Acknowledgments

The authors thank Adriano Sampaio e Sousa whose feedback on our previous publication initiated this project. The e-mail correspondence between A. S. e Sousa and Christopher L. Mulliss was very enjoyable and constructive. Wei-Da Chen acknowledges Chung Yuan Christian University for kindly offering to him an undergraduate research grant.

APPENDIX: WHY MULTIPLICATION AND DIVISION HAVE DIFFERENT LEVELS OF ACCURACY FOR THE CLM ROUNDING RULE

The statistical results for the CLM rounding rule, as revealed in Table VI, show a significantly distinct level of accuracy for multiplication and division. The detailed mathematical investigation of the CLM rule presented in this study led to an expression for the number of significant figures in the result in terms of the values of the leading digits. This reformulation, in turn, provides the key to explaining the mystery of why the CLM rule works better for division than for multiplication.

Eqs. (9) and (10) involve the leading digits of the numbers used in the calculation, a and a', and the leading digit of the result, a_x. Throughout this paper (e.g. Table V), it has been assumed that the value of a_x is statistically independent of the values of a and a'. The design of the Monte-Carlo simulation used in this paper ensures that a and a' are uniformly random. This assumption of independence would require that the value of a_x is also uniformly random. However we know this assumption is realistically incorrect for multiplication, because of the empirical observation (called Benford's Law) that the multiplication of (many) random numbers produces a result whose leading digit (d) is much more likely to be a small number (30% chance that $d = 1$) than a large number (5% chance that $d = 9$).

In order to determine if different frequency distributions of a_x exist for multiplication and division and how these distributions change when constraints (e.g. $a < 5$ or $a \geq 5$) are imposed, the following test was conducted. The values of y and z were allowed to range from 1 to 99. It was assumed that $N' > N$ and that y had the fewer number of significant digits (and therefore corresponded to a). For each combination of y and z, the following operations were performed and the leading digit of the result (a_x) recorded:

1. $x = y \cdot z$
2. $x = y/z$ (number with fewer significant figures in numerator)
3. $x = z/y$ (number with fewer significant figures in denominator)

TABLE VII: The frequency distribution of a_x for multiplication.

	Digit frequency (%) for multiplication ($x = y \cdot z$)		
Value of a_x	$1 \le a \le 9$	$a < 5$	$a \ge 5$
1	24.0%	34.6%	15.5%
2	18.3%	21.8%	15.6%
3	14.5%	13.3%	15.6%
4	11.8%	7.0%	15.7%
5	9.4%	4.5%	13.4%
6	7.7%	4.7%	10.1%
7	6.1%	4.7%	7.1%
8	5.6%	4.7%	4.6%
9	3.5%	4.8%	2.5%

TABLE VIII: The frequency distribution of a_x for division, where the number with fewer significant figures is in the numerator.

	Digit frequency (%) for division ($x = y/z$)		
Value of a_x	$1 \le a \le 9$	$a < 5$	$a \ge 5$
1	33.6%	23.6%	41.5%
2	14.9%	16.3%	13.8%
3	10.2%	14.2%	6.9%
4	8.3%	13.4%	4.1%
5	7.5%	11.2%	4.6%
6	6.9%	7.8%	6.2%
7	6.5%	5.7%	7.1%
8	6.4%	4.5%	8.0%
9	5.7%	3.2%	7.8%

For each of the operations listed above, the frequency distribution of a_x was computed under the following conditions:

A. $1 \le a \le 9$ (no constraints on a, included for comparison),

B. $a < 5$ (appropriate for the CLM rule),

C. $a \ge 5$ (appropriate for the CLM rule).

The resulting frequency distributions of a_x are presented in Tables VII, VIII, and IX. It is very clear from Tables VII and VIII that the frequency distributions of a_x are substantially different for multiplication and division (with no constraints on a). It is also very clear from Tables VII, VIII, and IX that the frequency distributions of a_x change dramatically under constraints (e.g. $a < 5$ or $a \ge 5$), and that this change is apparently different for multiplication and division. These facts point to the different statistical relationships between a_x and a as a possible explanation for the distinct levels of accuracy in the CLM rounding rule.

TABLE IX: The frequency distribution of a_x for division, where the number with fewer significant figures is in the denominator.

	Digit frequency (%) for division ($x = y/z$)		
Value of a_x	$1 \leq a \leq 9$	$a < 5$	$a \geq 5$
1	33.6%	32.3%	34.6%
2	14.9%	23.3%	8.2%
3	10.2%	12.6%	8.2%
4	8.3%	8.5%	8.0%
5	7.5%	6.4%	8.4%
6	6.9%	5.3%	8.3%
7	6.5%	4.4%	8.2%
8	6.4%	4.1%	8.3%
9	5.7%	3.2%	7.7%

An examination of Eq. (9), $N_x = N + \log(a/a_x)$, confirms that the dissimilar statistical relationships between a_x and a for multiplication and division explain the distinct levels of accuracy observed for the CLM rounding rule. According to Eq. (9), large values of a and/or small values of a_x are in favor of N_x to be equal to $N+1$. Similarly, small values of a and/or large values of a_x tend to cause $N_x = N$. The fact that division is more accurate than multiplication for the CLM rounding rule can now be examined.

For multiplication, when a is larger ($a \geq 5$), the value of a_x is weighted towards *higher* values. These two effects work against each other, making it somewhat less likely that $N_x = N + 1$ is the proper answer (as predicted by the CLM rule). This increase towards higher values of a_x for $a \geq 5$ does not happen for division; this is the reason why the CLM rule is significantly less accurate for multiplication.

For division, there are two operations to consider. One is when the number with the fewest significant figures is in the numerator, and the other when it is the denominator. In the first operation, the value of a_x is weighted dramatically towards the *lowest possible value* ($a_x = 1$) when a is larger ($a \geq 5$). These two effects work together to make it much more likely that $N_x = N + 1$ (as predicted by the CLM rule). In the second operation, the distribution of a_x values remains more or less unchanged when a is larger ($a \geq 5$). There is a slightly higher probability that $a_x = 1$, but the probability distribution for other values flattens out. With no substantial change in the distribution of the a_x values, a larger value of a ($a \geq 5$) makes it more likely that $N_x = N + 1$ is the proper answer (as predicted by the CLM rule).

In general, there is a statistical relationship between the value of a and the distribution of the a_x values. When a constraint (e.g. $a \geq 5$ or $a < 5$) is imposed on the numbers used in the calculation, there is a change (different for multiplication and division) in the frequency distribution of a_x. In short, the leading digits a and a_x in Eq. (9) are not independent of each other. There is a statistical relationship between them that is different for multiplication and division, and it depends on the constraints imposed on a. Fundamentally, this explains why the CLM rule works better for division than it does for multiplication.

References

* Electronic address: wlee@phys.cycu.edu.tw

[1] R. H. Good, Phys. Teach. **34**, 192 (1996).

[2] L. M. Schwartz, J. Chem. Edu. **62**, 693 (1985).

[3] B. L. Earl, J. Chem. Edu. **65**, 186 (1988).

[4] S. Stieg, J. Chem. Edu. **64**, 471 (1987).

[5] C. L. Mulliss and W. Lee, Chin. J. Phys. **36**, 479 (1998).

[6] A. S. e Sousa, private communication (2003).

[7] J. R. Taylor, *An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurement*, 2nd ed. (University Science Books, Sausalito, C. A., 1997), pp. 30–31.

[8] B. M. Shchigolev, *Mathematical Analysis of Observations* (Iliffe Books, London, 1965), p. 22.

[9] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, New York, 1992), p. 5.

[10] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. (Cambridge University Press, New York, 1992), pp. 272–273.