The Hodge conjecture

by
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1. Statement.

We recall that a pseudo complex structure on a C^∞-manifold X of dimension $2N$ is a $\mathbb{C}$-module structure on the tangent bundle T_X. Such a module structure induces an action of the group $\mathbb{C}^*$ on T_X, with $\lambda \in \mathbb{C}^*$ acting by multiplication by λ. By transport of structures, the group $\mathbb{C}^*$ acts also on each exterior power $\overset{n}{\wedge} T_X$, as well as on the complex dual $\Omega^n := \underline{\mathrm{Hom}}(\overset{n}{\wedge} T_X, \mathbb{C})$. For $p+q=n$, a (p,q)*-form* is a section of Ω^n on which $\lambda \in \mathbb{C}^*$ acts by multiplication by $\lambda^{-p}\bar{\lambda}^{-q}$.

From now on, we assume X complex analytic. A (p,q)-form is then a form which, in local holomorphic coordinates, can be written as

$$\sum a_{i_1,\ldots,i_p,j_1\ldots j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q},$$

and the decomposition $\Omega^n = \oplus \Omega^{p,q}$ induces a decomposition $d = d' + d''$ of the exterior differential, with d' (resp. d'') of degree $(1,0)$ (resp. $(0,1)$).

If X is compact and admits a Kähler metric, for instance if X is a projective non singular algebraic variety, this action of $\mathbb{C}^*$ on forms induces an action on cohomology. More precisely, $H^n(X,\mathbb{C})$ is the space of closed n-forms modulo exact forms, and if we define $H^{p,q}$ to be the space of closed (p,q)-forms modulo the $d'd''$ of $(p-1,q-1)$-forms, the natural map

$$\bigoplus_{p+q=n} H^{p,q} \to H^n(X,\mathbb{C}) \tag{1.1}$$

is an isomorphism. If we choose a Kähler structure on X, one can give the following interpretation to the decomposition (1.1) of $H^n(X,\mathbb{C})$: the action of $\mathbb{C}^*$ on forms commutes with the Laplace operator, hence induces an action of $\mathbb{C}^*$ on the space $\mathcal{H}^n$ of harmonic n-forms. We have $\mathcal{H}^n \xrightarrow{\sim} H^n(X,\mathbb{C})$ and $H^{p,q}$ identifies with the space of harmonic (p,q)-forms.

When X moves in a holomorphic family, the *Hodge filtration* $F^p := \bigoplus_{a \geq p} H^{a,n-a}$ of $H^n(X, \mathbb{C})$ is better behaved than the Hodge decomposition. Locally on the parameter space T, $H^n(X_t, \mathbb{C})$ is independent of $t \in T$ and the Hodge filtration can be viewed as a variable filtration $F(t)$ on a fixed vector space. It varies holomorphically with t, and obeys Griffiths transversality: at first order around $t_0 \in T$, $F^p(t)$ remains in $F^{p-1}(t_0)$.

So far, we have computed cohomology using C^∞ forms. We could as well have used forms with generalized functions coefficients, that is, currents. The resulting groups $H^n(X, \mathbb{C})$ and $H^{p,q}$ are the same. If Z is a closed analytic subspace of X, of complex codimension p, Z is an integral cycle and, by Poincaré duality, defines a class $\mathrm{cl}(Z)$ in $H^{2p}(X, \mathbb{Z})$. The integration current on Z is a closed (p,p)-form with generalized function coefficients, representing the image of $\mathrm{cl}(Z)$ in $H^{2p}(X, \mathbb{C})$. The class $\mathrm{cl}(Z)$ in $H^{2p}(X, \mathbb{Z})$ is hence of type (p,p), in the sense that its image in $H^{2p}(X, \mathbb{C})$ is. Rational (p,p)-classes are called *Hodge classes*. They form the group

$$H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X) = H^{2p}(X, \mathbb{Q}) \cap F^p \subset H^{2p}(X, \mathbb{C}).$$

In [6], Hodge posed the

Hodge conjecture. *On a projective non singular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes* $\mathrm{cl}(Z)$ *of algebraic cycles.*

2. Remarks.

(i) By Chow's theorem, on a complex projective variety, algebraic cycles are the same as closed analytic subspaces.

(ii) On a projective non singular variety X over $\mathbb{C}$, the group of integral linear combinations of classes $\mathrm{cl}(Z)$ of algebraic cycles coincides with the group of integral linear combinations of products of Chern classes of algebraic (equivalently by GAGA: analytic) vector bundles. To express $\mathrm{cl}(Z)$ in terms of Chern classes, one resolves the structural sheaf $\mathcal{O}_Z$ by a finite complex of vector bundles. That Chern classes are algebraic cycles holds, basically, because vector bundles have plenty of meromorphic sections.

(iii) A particular case of (ii) is that the integral linear combinations of classes of divisors (= codimension one cycles) are simply the first Chern classes of line bundles. If $Z^+ - Z^-$ is the divisor of a meromorphic section of $\mathcal{L}$, $c_1(\mathcal{L}) = \mathrm{cl}(Z^+) - \mathrm{cl}(Z^-)$. This is the starting point of the proof given by Kodaira and Spencer [7] of the Hodge conjecture for H^2: a class

$c \in H^2(X,\mathbb{Z})$ of type $(1,1)$ has image 0 in the quotient $H^{0,2} = H^2(X,\mathcal{O})$ of $H^2(X,\mathbb{C})$, and the long exact of cohomology defined by the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i\ \ )} \mathcal{O}^* \longrightarrow 0$$

shows that c is the first Chern class of a line bundle.

(iv) The relation between algebraic cycles and algebraic vector bundles is also the basis of Atiyah and Hirzebruch theorem [2] that the Hodge conjecture cannot hold integrally. In the Atiyah-Hirzebruch spectral sequence from cohomology to topological K-theory:

$$E_2^{pq} = H^p(X, K^q(P^t)) \Longrightarrow K^{p+q}(X),$$

the resulting filtration of $K^n(X)$ is by the

$$F^pK^n(X) = \mathrm{Ker}(K^n(X) \to K^n((p-1)\text{-skeleton, in any triangulation})).$$

Equivalently, a class c is in F^p if for some topological subspace Y of codimension p, c is the image of a class $\tilde{c}$ with support in Y. If Z is an algebraic cycle of codimension p, a resolution of $\mathcal{O}_Z$ defines a K-theory class with support in Z: $c_Z \in K^0(X, X-Z)$. Its image in $F^pK^0(X)$ agrees with the class of Z in $H^{2p}(X,\mathbb{Z})$. The later hence is in the kernel of the successive differentials d_r of the spectral sequence.

No cycle in $H^{2p}(X,\mathbb{Z})$, killed by all d_r and of type (p,p), is known not to be an integral linear combination of class $\mathrm{cl}(Z)$. One has no idea of which classes should be effective, that is, of the form $\mathrm{cl}(Z)$, rather than a difference of such.

On a Stein manifold X, any topological complex vector bundle can be given an holomorphic structure and, at least for X of the homotopy type of a finite CW complex, it follows that any class in $H^{2p}(X,\mathbb{Z})$ in the kernel of all d_r is a $\mathbb{Z}$-linear combination of classes of analytic cycles.

(v) The assumption in the Hodge conjecture that X be algebraic cannot be weakened to X being merely Kähler. See Zucker's appendix to [11] for counterexamples where X is a complex torus.

(vi) When Hodge formulated his conjecture, he had not realized it could hold only rationally. He also proposed a further conjecture, characterizing the subspace of $H^n(X,\mathbb{Z})$ spanned by the images of cohomology classes with support in a suitable closed analytic subspace of complex codimension k. Grothendieck observed that this further conjecture was trivially false, and gave a corrected version of it in [5].

3. The intermediate jacobian.

The cohomology class of an algebraic cycle Z of codimension p has a natural lift to a group $J_p(X)$, extension of the group of classes of type (p,p) in $H^{2p}(X,\mathbb{Z})$ by the *intermediate jacobian*

$$J_p(X)^0 := H^{2p-1}(X,\mathbb{Z}) \setminus H^{2p-1}(X,\mathbb{C})/F^p.$$

This expresses that the class can be given an integral description (in singular cohomology), as well as an analytic one, as a closed (p,p) current, giving a hypercohomology class in $\mathbb{H}^{2p}$ of the subcomplex $F^p\Omega^*_{\text{hol}} := (0 \to \cdots \to 0 \to \Omega^p_{\text{hol}} \to \cdots)$ of the holomorphic de Rham complex, with an understanding at the cocycle level of why the two agree in $H^{2p}(X,\mathbb{C})$. Indeed, $J_p(X)$ is the hypercohomology $\mathbb{H}^{2p}$ of the homotopy kernel of the difference map $\mathbb{Z} \oplus F^p\Omega^*_{\text{hol}} \to \Omega^*$.

In general, using that all algebraic cycles on X fit in a denumerable number of algebraic families, one checks that the subgroup $A_p(X)$ of $J_p(X)$ generated by algebraic cycles is the extension of a denumerable group by its connected component $A^0_p(X)$, and that for some sub-Hodge structure H_{alg} of type $\{(p-1,p),(p,p-1)\}$ of $H^{2p-1}(X)$, $A^0_p(X)$ is $H_{\text{alg}_\mathbb{Z}} \setminus H_{\text{alg}_\mathbb{C}}/F^p$. "Sub-Hodge structure" means: subgroup of the integral lattice, with complexification sum of its intersections with the $H^{a,b}$. The Hodge conjecture (applied to the product of X and a suitable abelian variety) predicts that H_{alg} is the biggest sub-Hodge structure of $H^{2p-1}(X)$ of type $\{(p-1,p)(p,p-1)\}$.

No conjecture is available to predict what subgroup of $J_p(X)$ the group $A_p(X)$ is. Cases are known where $A_p(X)/A^0_p(X)$ is of infinite rank. See for instance the recent paper [9] and the references it contains. This has made generally inapplicable the methods introduced by Griffiths (see, for instance, Zucker [11]) to prove the Hodge conjecture by induction on the dimension of X, using a Lefschetz pencil of hyperplane sections of X. Indeed, the method requires not just the Hodge conjecture for the hyperplane sections H, but that all of $J_p(H)$ comes from algebraic cycles.

4. Detecting Hodge classes.

Let $(X_s)_{s\in S}$ be an algebraic family of projective non singular algebraic varieties: the fibers of a projective and smooth map $f\colon X \to S$. We assume it is defined over the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. No algorithm is known to decide whether a given integral cohomology class of a typical fiber X_0 is somewhere on S of type (p,p). The Hodge conjecture implies that the locus where this happens is a denumerable union of algebraic subvarieties of S (known: see [4]), and is defined over $\bar{\mathbb{Q}}$ (unknown).

Even if in general it is difficult to detect Hodge classes, there are cases of integral cycles which are clearly (p,p), and for which the Hodge conjecture is particularly tantalizing.

Example 1. For X of complex dimension N, the diagonal Δ of $X \times X$ is an algebraic cycle of codimension N. The Hodge decomposition being compatible with Künneth, the Künneth components $\mathrm{cl}(\Delta)_{a,b} \in H^a(X) \otimes H^b(X) \subset H^{2N}(X \times X)$ $(a+b=2N)$ of $\mathrm{cl}(\Delta)$ are Hodge classes.

Example 2. If $\eta \in H^2(X,\mathbb{Z})$ is the cohomology class of an hyperplane section of X, the iterated cup product $\eta^p\colon H^{N-p}(X,\mathbb{C}) \to H^{N+p}(X,\mathbb{C})$ is an isomorphism (hard Lefschetz theorem, proved by Hodge. See [10] IV.6). Let $\mathfrak{z} \in H^{N-p}(X,\mathbb{C}) \otimes H^{N-p}(X,\mathbb{C}) \subset H^{2N-2p}(X \times X)$ be the class such that the inverse isomorphism $(\eta^p)^{-1}$ is $c \mapsto \mathrm{pr}_{1!}(\mathfrak{z} \cup \mathrm{pr}_2^* c)$. The class $\mathfrak{z}$ is Hodge.

5. Motives.

Algebraic varieties admit a panoply of cohomology theories, related over $\mathbb{C}$ by comparison isomorphisms. Resulting structures on $H^*(X,\mathbb{Z})$ should be viewed as analogous to the Hodge structure. Examples: if X is defined over a subfield K of $\mathbb{C}$, with algebraic closure $\bar{K}$ in $\mathbb{C}$, $\mathrm{Gal}(\bar{K}/K)$ acts on $H^*(X,\mathbb{Z}) \otimes \mathbb{Z}_\ell$ and $H^*(X,\mathbb{C}) = H^*(X,\mathbb{Z}) \otimes \mathbb{C}$ has a natural K-structure $H_{\mathrm{DR}}(X \text{ over } K)$, compatible with the Hodge filtration. Those cohomology theories give rise to conjectures parallel to the Hodge conjecture, determining the linear span of classes of algebraic cycles. Example: the Tate conjecture [8]. Those conjectures are open even for H^2.

Grothendieck's theory of motives aims at understanding the parallelism between those cohomology theories. Progress is blocked by a lack of methods to construct interesting algebraic cycles. If the cycles of examples 1 and 2 of §4 were algebraic, Grothendieck's motives over $\mathbb{C}$ would form a semi-simple abelian category with a tensor product, and be the category of representations of some pro-reductive group-scheme. If the algebraicity of those cycles is assumed, the full Hodge conjecture is equivalent to a natural functor from the category of motives to the category of Hodge structures being fully faithful.

6. Substitutes and weakened forms

In despair, efforts have been made to find substitutes for the Hodge conjecture. On abelian varieties, Hodge classes at least share many properties of cohomology classes of algebraic cycles: they are "absolutely Hodge" ([3]), even "motivated" ([1]). This suffices for some applications – for instance the proof of algebraic relations among periods and quasi periods of abelian varieties predicted by the Hodge conjecture ([3]), but does not allow for reduction modulo p. The following corollary of the Hodge conjecture would be particularly interesting. Let A be an abelian variety over the algebraic closure $\mathbb{F}$ of a finite field $\mathbb{F}_q$. Lift A in two different ways to characteristic 0, to complex abelian varieties A_1 and A_2 defined over $\bar{\mathbb{Q}}$. Pick Hodge classes $\mathfrak{z}_1$ and $\mathfrak{z}_2$ on A_1 and A_2, of complementary dimension. Interpreting $\mathfrak{z}_1$ and $\mathfrak{z}_2$ as ℓ-adic cohomology classes, one can define the intersection number κ of the reduction of $\mathfrak{z}_1$ and $\mathfrak{z}_2$ over $\mathbb{F}$. Is κ a rational number? If $\mathfrak{z}_1$ and $\mathfrak{z}_2$ were $\mathrm{cl}(Z_1)$ and $\mathrm{cl}(Z_2)$, Z_1 and Z_2 could be chosen to be defined over $\bar{\mathbb{Q}}$ and κ would be the intersection number of the reductions of Z_1 and Z_2.

References

[1] Y. André, Pour une théorie inconditionnelle des motifs, Publ. Math. IHES **83** (1996) p. 5–49.

[2] M. F. Atiyah and F. Hirzebruch, Analytic cycles on complex manifolds, Topology **1** (1962) p. 25–45.

[3] P. Deligne (rédigé par J. L. Brylinski), Cycles de Hodge absolu et périodes des intégrales des variétés abéliennes, Mémoires SMF **2** (1980) p. 23–33.

[4] P. Deligne, E. Cattani and A. Kaplan, On the locus of Hodge classes, JAMS **8** 2 (1995) p. 483–505.

[5] A. Grothendieck, Hodge's general conjecture is false for trivial reasons, Topology **8** (1969) p. 299–303.

[6] W. V. D. Hodge, The topological invariants of algebraic varieties, Proc. ICM 1950, p. 181–192.

[7] K. Kodaira and D. C. Spencer, Divisor classes on algebraic varieties, Proc. Nat. Acad. Sci. **39** (1953) p. 872–877.

[8] J. Tate, Algebraic cycles and poles of zeta functions, Arithmetic Algebraic Geometry, Harper and Row, New York 1965, p. 93–110.

[9] C. Voisin, The Griffiths group of a General Calabi-Yau threefold is not finitely generated, Duke Math. J. **102** 1, p. 151–186.

[10] A. Weil, Introduction à l'étude des Variétés kahlériennes, Publ. Univ. Nancago VI, Act. Sci. et Ind. **1267**, Hermann, Paris 1958.

[11] S. Zucker, The Hodge Conjecture for cubic fourfolds, Comp. Math. **34** 2 (1977) p. 199–209.