

## The Poincaré Conjecture

In 1904, Henri Poincaré [17, pp. 486, 498] asked the following question. “*Considérons maintenant une variété [fermée]  $V$  à trois dimensions  $\cdots$  Est-il possible que le groupe fondamental de  $V$  se réduise à la substitution identique, et que pourtant  $V$  ne soit pas simplement connexe?*” Translated both into English and into more modern terminology, this becomes:

*Consider a compact 3-dimensional manifold  $V$  without boundary. Is it possible that the fundamental group of  $V$  could be trivial, even though  $V$  is not homeomorphic to the 3-dimensional sphere?*

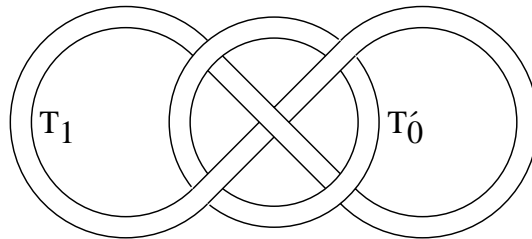
He commented, with considerable foresight, “*Mais cette question nous entraînerait trop loin*”. Since then, the hypothesis that every simply connected closed 3-manifold is homeomorphic to the 3-sphere has been known as the *Poincaré conjecture*. It has inspired topologists ever since, leading to many false proofs, but also to many advances in our understanding of the topology of manifolds.

### Early Missteps.

Four years earlier, Poincaré [17, p. 370] had stated the following false theorem “*dont la démonstration demanderait quelques développements*”.

*Every compact polyhedral manifold with the homology of an  $n$ -dimensional sphere is actually homeomorphic to the  $n$ -dimensional sphere.*

(Again, I have restated his assertion in more modern language.) However, by 1904 he had developed the concept of fundamental group, and constructed a beautiful counterexample to this statement. His example can be described as the coset space  $M^3 = \text{SO}(3)/I_{60}$  where  $I_{60}$  is the group of rotations which carry a regular icosahedron onto itself. (In other words,  $M^3$  can be identified with the space whose elements are regular icosahedra of unit diameter centered at the origin in 3-space.) This space has a non-trivial fundamental group  $\pi_1(M^3)$  of order 120.



The next important false theorem was by Henry Whitehead in 1934. As part of a purported proof of the Poincaré conjecture, he claimed that every contractible open 3-dimensional manifold is homeomorphic to Euclidean space. Following in Poincaré’s footsteps, he then discovered a counterexample to his own theorem, and thus substantially increased our understanding of the topology of manifolds. (See [26, pp. 21-50].) His counterexample can be briefly described as follows. Start with two disjoint solid tori  $T_1$  and  $T'_0$  in the 3-sphere which are embedded as shown, with linking number zero. Since  $T'_0$  is unknotted, its complement  $T_0 = S^3 \setminus \text{interior}(T'_0)$  is also a solid torus, with  $T_0 \supset T_1$ , but with  $\pi_1(T_0 \setminus T_1)$  nonabelian. Choose a homeomorphism  $h$  of the 3-sphere which maps  $T_0$  onto

$T_1$ . We then inductively construct closed unknotted solid tori

$$\cdots \supset T_{-1} \supset T_0 \supset T_1 \supset T_2 \supset \cdots$$

in  $S^3$ , where  $T_{n+1} = h(T_n)$ . The complement  $S^3 \setminus \bigcap T_n$  (or the union  $\bigcup T_n$ ) is the required Whitehead counterexample, a contractible manifold which is not simply connected at infinity.

For a delightful presentation of some of the further pitfalls of 3-dimensional topology, see Bing. For a representative collection of attacks on the Poincaré conjecture, see the papers by Birman, Gabai, Gillman and Rolfsen, Jakobsche, Papakyriakopoulos, Rourke, and Thickstun, as listed below.

### Higher Dimensions.

The late 1950's and early 1960's saw an avalanche of progress with the discovery that higher dimensional manifolds are actually easier to work with than 3-dimensional ones. (One reason for this is the following: The fundamental group plays an important role in all dimensions even when it is trivial, and relations in the fundamental group correspond to 2-dimensional disks, mapped into the manifold. In dimension 5 or more, such disks can be put into general position so that there are no self-intersections, but in dimension 3 or 4 it may not be possible to avoid self-intersections, leading to serious difficulties.)

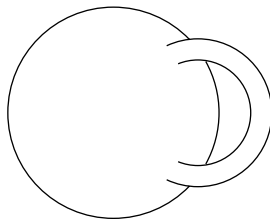
Stephen Smale announced a proof of the Poincaré conjecture in high dimensions in 1960. He was quickly followed by John Stallings, who used a completely different method, and by Andrew Wallace, who had been working along lines quite similar to those of Smale.

Let me first describe the Stallings result, which has a weaker hypothesis and easier proof, but also a weaker conclusion. He assumed that the dimension is seven or more, but Zeeman later extended his argument to dimensions five and six.

**Stallings-Zeeman Theorem.** *If  $M^n$  is a finite simplicial complex of dimension  $n \geq 5$  which has the homotopy type of the sphere  $S^n$  and is locally piecewise linearly homeomorphic to the Euclidean space  $\mathbb{R}^n$ , then  $M^n$  is homeomorphic to  $S^n$  under a homeomorphism which is piecewise linear except at a single point. In other words, the complement  $M^n \setminus (\text{point})$  is piecewise linearly homeomorphic to  $\mathbb{R}^n$ .*

(The method of proof consists of pushing all of the difficulties off towards a single point, so that there can be no control near that point.)

The Smale proof, and the closely related proof given shortly afterwards by Wallace, depended rather on differentiable methods, building a manifold up inductively, starting with an  $n$ -dimensional ball, by successively adding handles. Here a  $k$ -*handle* can be added to a manifold  $M^n$  with boundary by first attaching a  $k$ -dimensional cell, using an attaching homeomorphism from the  $(k-1)$ -dimensional boundary sphere into the boundary of  $M^n$ , and then thickening and smoothing corners so as to obtain a larger manifold with boundary. The proof is carried out by rearranging and cancelling such handles. (Compare the presentation in [13].)



3-dimensional ball with a 1-handle attached.

**Smale Theorem.** *If  $M^n$  is a differentiable homotopy sphere of dimension  $n \geq 5$ , then  $M^n$  is homeomorphic to  $S^n$ . In fact  $M^n$  is diffeomorphic to a manifold obtained by gluing together the boundaries of two closed  $n$ -balls under a suitable diffeomorphism.*

This was also proved by Wallace, at least for  $n \geq 6$ . (It should be noted that the 5-dimensional case is particularly difficult.)

The much more difficult 4-dimensional case had to wait twenty years, for the work of Michael Freedman. Here the differentiable methods used by Smale and Wallace and the piecewise linear methods used by Stallings and Zeeman do not work at all. Freedman used wildly non-differentiable methods, not only to prove the 4-dimensional Poincaré conjecture, but also to give a complete classification of closed simply connected topological 4-manifolds. The integral cohomology group  $H^2$  of such a manifold is free abelian. Freedman needed just two invariants: The cup product  $\beta : H^2 \otimes H^2 \rightarrow H^4 \cong \mathbb{Z}$  is a symmetric bilinear form with determinant  $\pm 1$ , while the *Kirby-Siebenmann invariant*  $\kappa$  is an integer mod 2 which vanishes if and only if the product manifold  $M^4 \times \mathbb{R}$  can be given a differentiable structure.

**Freedman Theorem.** *Two closed simply connected 4-manifolds are homeomorphic if and only if they have the same bilinear form  $\beta$  and the same Kirby-Siebenmann invariant  $\kappa$ . Any  $\beta$  can be realized by such a manifold. If  $\beta(x \otimes x)$  is odd for some  $x \in H^2$ , then either value of  $\kappa$  can be realized also. However, if  $\beta(x \otimes x)$  is always even, then  $\kappa$  is determined by  $\beta$ , being congruent to one eighth of the signature of  $\beta$ .*

In particular, if  $M^4$  is a homotopy sphere, then  $H^2 = 0$  and  $\kappa = 0$ , so  $M^4$  is homeomorphic to  $S^4$ . It should be noted that the piecewise linear or differentiable theories in dimension 4 are much more difficult. In particular, it is not known which 4-manifolds with  $\kappa = 0$  actually possess differentiable structures, and it is not known when this structure is essentially unique. The major results on these questions are due to Donaldson. As one indication of the complications, Freedman showed, using Donaldson's work, that  $\mathbb{R}^4$  admits uncountably many inequivalent differentiable structures. (Compare Gompf.)

### The Thurston Program.

In dimension three, the discrepancies between topological, piecewise linear, and differentiable theories disappear (see Hirsch, Munkres, Moise), but difficulties with the fundamental group become severe.

A far reaching conjecture by Thurston holds that every three manifold can be cut up along 2-spheres and tori so as to decompose into essentially unique pieces, each of which has a simple geometrical structure. There are eight 3-dimensional geometries in Thurston's program. Six of these are now well understood, and there has been a great deal of progress with

the geometry of constant negative curvature. However, the eighth geometry, corresponding to constant positive curvature, remains largely untouched. For this geometry, we have the following extension of the Poincaré conjecture.

**Thurston Elliptization Conjecture.** *Every closed 3-manifold with finite fundamental group has a metric of constant positive curvature, and hence is homeomorphic to a quotient  $S^3/\Gamma$ , where  $\Gamma \subset \text{SO}(4)$  is a finite group of rotations which acts freely on  $S^3$ .*

(The Poincaré conjecture corresponds to the special case where  $\Gamma \cong \pi_1(M^3)$  is trivial.) The possible subgroups  $\Gamma \subset \text{SO}(4)$  were classified long ago by Hopf, but this conjecture remains wide open. (For references up to 1995 on the Thurston Program, see [12, p. 93]. For a recent attack, see [1].)

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