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On Hamiltonian cycles and Hamiltonian paths

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Abstract

A Hamiltonian cycle is a spanning cycle in a graph, i.e., a cycle through every vertex, and a Hamiltonian path is a spanning path. In this paper we present two theorems stating sufficient conditions for a graph to possess Hamiltonian cycles and Hamiltonian paths. The significance of the theorems is discussed, and it is shown that the famous Ore's theorem directly follows from our result.

Keywords: Combinatorial problems; Graphs; Hamiltonian cycle; Hamiltonian path

1. Introduction

We consider only simple graphs, i.e., graphs with no multi-edges and no self loops, and every reference to a cycle or a path, unless otherwise specified, indicates, respectively, a simple cycle or a simple path.

A Hamiltonian cycle is a spanning cycle in a graph, i.e., a cycle through every vertex and a Hamiltonian path is a spanning path. A graph containing a Hamiltonian cycle is said to be Hamiltonian. It is clear that every graph with a Hamiltonian cycle has a Hamiltonian path but the converse is not necessarily true. The study of Hamiltonian cycles and Hamiltonian paths in general and special graphs has been motivated by practical applications and by the issues of complexity. The problem of finding whether a graph G is Hamiltonian is proved to be NP-complete for general graphs [4]. The problem remains NP-complete (see [4])

(1) if G is planar, cubic, 3-connected, and has no face with fewer than 5 edges,
(2) if G is bipartite,
(3) if G is the square of a graph,
(4) if a Hamiltonian path for G is given as part of the instance.

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On the other hand the problem of finding whether a graph G contains a Hamiltonian path is also proved to be NP-complete for general graphs [4]. Again, it remains NP-complete

(1) if G is planar, cubic, 3-connected, and has no face with fewer than 5 edges,
(2) if G is bipartite.

Even the variant in which either the starting point or the end point or both are specified in the input instance is also NP-complete. No easily testable characterization is known for Hamiltonian graphs. Nor there exists any such condition to test whether a graph contains a Hamiltonian path or not. Although research efforts have been spent for finding the necessary and sufficient conditions for a graph to be Hamiltonian, the resulting conditions have proved to be hardly useful from the practical algorithmic context (which is obvious due to the NP-completeness results). For example, Plotnikov in [8] presented a necessary and sufficient condition for hamiltonicity which involves the number of elements in any independent set (an *independent set* in a graph is a vertex subset that contains no edge in that graph) of a graph. Since the number of independent sets of a graph is a value of the order of $O(2^n)$, the direct use of the offered criterion is difficult except for further theoretic research. A number of researchers have also investigated various other relations and structural properties of Hamiltonian graphs [5,6,9]. On the other hand, tremendous amount of research has been done in finding the sufficient conditions for the existence of Hamiltonian cycles or Hamiltonian paths in graphs. Before presenting some of the conditions in the literature we need to introduce and define some of the notations. Given a graph $G = (V, E)$ and a vertex $u \in V$, we denote by $d(u)$ the degree of u in G. In other words, $d(u) = |N_G(u)|$, where $N_G(u)$ denotes the neighbor set of u in a graph G. If $H \subseteq G$ then $d_H(u) = |N_H(u)|$ and $d_{\overline{H}}(u) = |N_{G\setminus H}(u)|$. If $P \equiv \langle u = u_0, u_2, \ldots, u_k = v\rangle$ is a path in G, then the length of the path P is k, i.e., the number of edges in P. By $\delta(u, v)$, we denote the length of a shortest path between u and v in G. On the other hand by $\delta(G)$ we indicate the minimum of vertex degrees in G. $V[G]$ and $E[G]$ are used to denote, respectively, the vertex set and edge set of G.

Now we are ready to list some of the conditions available in the literature for the existence of Hamiltonian cycles or paths in graphs. In particular, below we list some of the conditions that are related to our results.

Theorem 1.1 (Dirac [3]). *If G is a simple graph with n vertices where $n \geqslant 3$ and $\delta(G) \geqslant n/2$, then G is Hamiltonian.*

Theorem 1.2 (Ore [7]). *Let G be a simple graph with n vertices and u, v be distinct nonadjacent vertices of G with $d(u) + d(v) \geqslant n$. Then G is Hamiltonian if and only if $G + (u, v)$ is Hamiltonian.*

Theorem 1.3 (Bondy and Chvátal [1]). *If G is a simple graph with n vertices, then G is Hamiltonian if and only if its closure is Hamiltonian.*

Remark. The (Hamiltonian) closure of a graph G, denoted by $C(G)$, is the supergraph of G on $V(G)$ obtained by iteratively adding edges between pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. Fortunately, the closure does not depend on the order in which we choose to add edges when more than one is available, i.e., the closure of G is well-defined (for a proof of this statement see [10]).

Theorem 1.4 (Ore [7]). *If $d(u) + d(v) \geqslant n$ for every pair of distinct nonadjacent vertices u and v of G, then G is Hamiltonian.*

The main results of our paper are the following two theorems.

Theorem 1.5. *Let $G = (V, E)$ be a connected graph with n vertices and P be a longest path in G having length k and with end vertices u and v. Then the following statements must hold*:

(a) *Either $\delta(u, v) > 1$ or P is a Hamiltonian path contained in a Hamiltonian cycle.*
(b) *If $\delta(u, v) \geqslant 3$ then $d_p(u) + d_p(v) \leqslant k - \delta(u, v) + 2$.*
(c) *If $\delta(u, v) = 2$, then either $d_p(u) + d_p(v) \leqslant k$ or P is a Hamiltonian path and there is a Hamiltonian cycle.*

Theorem 1.6. *Let $G = (V, E)$ be a connected graph with n vertices such that for all pairs of distinct non-adjacent vertices $u, v \in V$ we have $d(u) + d(v) + \delta(u, v) \geqslant n + 1$. Then G has a Hamiltonian path.*

It will be shown in this paper that famous Ore's conditions, listed above (Theorems 1.2 and 1.4), directly follow from our results. Also the introduction of the parameter $\delta(u, v)$ in Theorem 1.6 seems to be significant with respect to the related degree conditions for Hamiltonian paths and cycles in graphs.

The rest of the paper is organized as follows. In Section 2 we present our main results. Section 3 establishes the significance of our results. We conclude in Section 4 by introducing some open problems for future research.

2. Main results

We begin this section by presenting the following lemma.

Lemma 2.1. *Let $G = (V, E)$ be a connected graph with n vertices and P be a longest path in G. If P is contained in a cycle then P is a Hamiltonian path.*

Proof. Suppose $P \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v\rangle$, and P is contained in a cycle $C \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v, u_0 = u\rangle$. Note that $V(C) = V(P)$, since otherwise P would be a part of a longer path, a contradiction. Assume for the sake of contradiction that $k < n - 1$, i.e., P is not Hamiltonian path. Since G is connected, there must be an edge of the form (x, y) such that $x \in V(P) = V(C)$ and $y \in V(G) - V(C)$. Let $x = u_i$. Then there is a path $P' \equiv \langle y, x = u_i, u_{i+1}, \ldots, u_k, u_0, u_1, u_2, \ldots, u_{i-1}\rangle$ with length $k + 1$, which is a contradiction, since P is a longest path in G. □

Corollary 2.2. *Let $G = (V, E)$ be a connected graph with n vertices and P be a longest path in G. If P is contained in a cycle then G is Hamiltonian.*

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. (a) Assume $\delta(u, v) \leqslant 1$. Since the graph is connected, $\delta(u, v) = 1$. Let $P \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v\rangle$. Since $\delta(u, v) = 1$, we in effect have a cycle $C \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v, u_0 = u\rangle$ and the result readily follows from Lemma 2.1.

(b) Assume $\delta(u, v) \geqslant 3$. In this case surely $d_p(u) + d_p(v) \leqslant k - \delta(u, v) + 2 \leqslant k - 3 + 2 = k - 1$, since otherwise we would get a path from u to v with length less than $\delta(u, v)$, a contradiction.

(c) Assume that $\delta(u, v) = 2$. Now note that we cannot claim that $d_p(u) + d_p(v) \leqslant k - \delta(u, v) + 2 = k$ by arguing contradiction on $\delta(u, v)$ as we did in (b) because now there is a common vertex adjacent to both u and v. However, we argue in a different way as follows. Assume that $d_p(u) + d_p(v) \geqslant k + 1 = |V(P)|$. We rewrite the path P as follows: $P \equiv \langle v = w_1, w_2, \ldots, w_{|V(P)|-1}, w_{|V(P)|} = u\rangle$. Now we will try to find out two crossover edges (v, w_{i+1}) and (w_i, u) such that we get the cycle $C = \langle w_1, w_{i+1}, w_{i+2}, \ldots, w_{|V(P)|-1}, w_{|V(P)|}, w_i, w_{i-1}, \ldots, w_2, w_1\rangle$. To see that this is possible, let us consider $S = \{j \mid (v, w_{j+1}) \in E\}$ and $T = \{j \mid (w_j, u) \in E\}$. Since $S \cup T \subseteq \{1, 2, \ldots, |V(P)| - 1\}$, we have $|S \cup T| \leqslant |V(P)| - 1$. Again because, $|S| = d_p(v)$, $|T| = d_p(u)$, and $d_p(u) + d_p(v) \geqslant |V(P)|$, we must have,

$$\begin{aligned} |S \cap T| &= |S| + |T| - |S \cup T| \\ &= d_p(u) + d_p(v) - |S \cup T| \\ &\geqslant d_p(u) + d_p(v) - \big(|V(P)| - 1\big) \\ &\geqslant \big|V(P)\big| - \big|V(P)\big| + 1 \\ &= 1. \end{aligned}$$

Hence S and T must have a common subscript so that the two crossover edges (v, w_{i+1}) and (w_i, u) exist and we get the cycle C. So, in effect, we get a cycle C, which contains a Hamiltonian path $P' = \langle w_1, w_{i+1}, w_{i+2}, \ldots, w_{|V(P)|-1}, w_{|V(P)|}, w_i, w_{i-1}, \ldots, w_2\rangle$ (slightly different from P). Hence by Lemma 2.1 the result follows. □

Corollary 2.3. *Let $G = (V, E)$ be a connected graph with n vertices, and P be a longest path in G having length $k < n - 1$ and with end vertices u and v. Then we must have $d_p(u) + d_p(v) \leqslant k - \delta(u, v) + 2$.*

Proof. Since $k < n - 1$, P is not a Hamiltonian path. Hence by Theorem 1.5(a) we have $\delta(u, v) > 1$. Noting that if $\delta(u, v) = 2$, then $k = k - \delta(u, v) + 2$, by Theorem 1.5(b) and (c) we thus have $d_p(u) + d_p(v) \leqslant k - \delta(u, v) + 2$. □

Now we are ready to give the proof of Theorem 1.6.

Proof of Theorem 1.6. We prove it by contradiction as follows. Assume that the condition holds but there is no Hamiltonian path in G. Then let $P \equiv \langle u = u_0, u_1, \ldots, u_k = v \rangle$ be a longest path in G. Surely, $k \leqslant n-2$ and $\delta(u,v) \leqslant k$. Then by Corollary 2.3 we must have $d_p(u) + d_p(v) \leqslant k - \delta(u,v) + 2$. Now we have,

$$\begin{aligned}
&d(u) + d(v) + \delta(u,v) \\
&\quad = d_p(u) + d_{\bar{p}}(u) + d_p(v) + d_{\bar{p}}(v) + \delta(u,v) \\
&\quad = \{d_p(u) + d_p(v)\} + d_{\bar{p}}(u) + d_{\bar{p}}(v) + \delta(u,v) \\
&\quad \leqslant \{k - \delta(u,v) + 2\} + d_{\bar{p}}(u) + d_{\bar{p}}(v) + \delta(u,v) \\
&\quad = k + 2 + d_{\bar{p}}(u) + d_{\bar{p}}(v) \\
&\quad \leqslant n - 2 + 2 + d_{\bar{p}}(u) + d_{\bar{p}}(v) \\
&\quad = n + d_{\bar{p}}(u) + d_{\bar{p}}(v).
\end{aligned}$$

Since P is not a Hamiltonian path, by Theorem 1.5(a), $\delta(u,v) > 1$, i.e., u, v are nonadjacent and hence we have $d(u) + d(v) + \delta(u,v) \geqslant n+1$ according to our assumption. We thus have,

$$\begin{aligned}
&n + d_{\bar{p}}(u) + d_{\bar{p}}(v) \geqslant n+1 \\
&\quad \Rightarrow d_{\bar{p}}(u) + d_{\bar{p}}(v) \geqslant n - n + 1 \\
&\quad \Rightarrow d_{\bar{p}}(u) + d_{\bar{p}}(v) \geqslant 1.
\end{aligned}$$

Hence there is at least one edge of the form (x,y) such that $x \in \{u,v\}$ and $y \in V(G) - V(P)$ which means that we get a longer path in G by adding the edge (x,y) to P which is a contradiction and the result follows. □

3. Significance of our results

A number of existing well known and very powerful theorems directly follow from our results as discussed below. Consider Theorem 1.1, i.e., Dirac's condition. The proof of Dirac's Theorem very cleverly exploits the idea of extremality. The idea is: if there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree, so we may restrict our attention to maximal non-Hamiltonian graphs with minimum degree at least $n/2$. By "maximal", we mean that no proper supergraph is also non-Hamiltonian, so $G + (u,v)$ is Hamiltonian whenever u, v are nonadjacent. Note that the maximality of G implies that G has a spanning path from $u = v_1$ to $v = v_n$, i.e., a Hamiltonian path. The rest of the proof tries to find a crossover to construct a spanning cycle [10]. The result provided by Ore (Theorem 1.2) is in fact inspired form Dirac's condition. Ore observed that this argument uses $\delta(G) \geqslant n/2$ only to show that $d(u) + d(v) \geqslant n$. Therefore, we can weaken the requirement of minimum degree $n/2$ and ask for $d(u) + d(v) \geqslant n$ whenever u, v are nonadjacent. We also do not need G to be a maximal non-Hamiltonian graph, only that $G + (u,v)$ is Hamiltonian and thereby provide a spanning u, v-path. We here show that Ore's conditions, in fact, follow from our results. First we present the following lemma.

Lemma 3.1. *Let G be a simple graph with n vertices and u, v are distinct nonadjacent vertices of G with $d(u) + d(v) \geqslant n$. Then $\delta(u,v) = 2$.*

Proof. Let us arrange the vertices of G in a sequence such that $V(G) = \{v = w_1, w_2, \ldots, w_{|V(P)|-1}, w_{|V(P)|} = u\}$. Let $S = \{j \mid (v, w_j) \in E\}$ and $T = \{j \mid (w_j, u) \in E\}$. Since $S \cup T \subseteq \{2, \ldots, |V(P)| - 1\}$, we have $|S \cup T| \leqslant |V(P)| - 2$. Again because, $|S| = d_p(v)$, $|T| = d_p(u)$, and $d_p(u) + d_p(v) \geqslant |V(P)|$, we must have,

$$\begin{aligned}
|S \cap T| &= |S| + |T| - |S \cup T| \\
&= d_p(u) + d_p(v) - |S \cup T| \\
&\geqslant d_p(u) + d_p(v) - (|V(P)| - 2) \\
&\geqslant |V(P)| - |V(P)| + 2 \\
&= 2.
\end{aligned}$$

Hence S and T must have common subscripts so that we have edges of the form (u,x), (x,v) which implies a u, v-path of length 2. Since u, v are nonadjacent the result follows. □

Now we are ready to prove that Ore's theorem (Theorem 1.2) follows from our result.

Proof of Ore's theorem (Theorem 1.2). One direction is trivial. So we prove the other direction as follows. By Lemma 3.1 since G satisfies the sufficient conditions, we must have $\delta(u,v) = 2$. Now since $G + (u,v)$ is Hamiltonian we must have a Hamiltonian path say P in G. So P is a longest path in G. Since

P is a Hamiltonian path, we have $d_P(u) + d_P(v) = d(u) + d(v) \geqslant n$. Hence by Theorem 1.5(c), P is contained in a Hamiltonian cycle and hence G is Hamiltonian. □

Now we will consider Theorem 1.4 which is also due to Ore. Here again we bank upon the following lemma.

Lemma 3.2. *Let G be a simple graph and $d(u) + d(v) \geqslant n$ for every pair of distinct nonadjacent vertices u and v of G. Then $\delta(u, v) \leqslant 2$ for every pair of distinct vertices u and v of G.*

Proof. First note that for every pair of distinct adjacent vertices $\delta(u, v) = 1 < 2$. Now we just need to consider every pair of distinct nonadjacent vertices. Then the result follows readily from Lemma 3.1. □

Now we are ready to show that Theorem 1.4 also follows from our result.

Proof of Ore's theorem (Theorem 1.4). Since we have $d(u) + d(v) \geqslant n$ for every pair of distinct nonadjacent vertices u and v, by Lemma 3.2, $\delta(u, v) \leqslant 2$ for every pair of distinct vertices u and v. And it is clear that for every such pair u and v we must have $\delta(u, v) = 2$. Now for every pair of distinct nonadjacent vertices u and v we have,

$$\begin{aligned} &d(u) + d(v) \geqslant n > n + 1 - 2 = n + 1 - \delta(u, v) \\ &\quad \Rightarrow d(u) + d(v) > n + 1 - \delta(u, v) \\ &\quad \Rightarrow d(u) + d(v) + \delta(u, v) > n + 1. \end{aligned}$$

Thus by Theorem 1.6 there is a Hamiltonian path P (let) in G. Now P is a longest path in G. Let the end vertices of P be x and y. If we have $\delta(x, y) = 1$ then by Theorem 1.5(a), P is contained in a Hamiltonian cycle and hence G is Hamiltonian. Otherwise we must have $\delta(x, y) = 2$. And since we have $d(u) + d(v) \geqslant n > n - 1$, by Theorem 1.5(b) again P is contained in a Hamiltonian cycle and hence G is Hamiltonian. □

Notice that the sufficient conditions of Theorem 1.6 can be checked in $O(n^3)$ with an all pair shortest path algorithm (for instance, the Floyd–Warshall algorithm [2]).

4. Conclusion and future works

In this paper we present two theorems (Theorems 1.5 and 1.6) and show how the famous Ore's theorems (Theorems 1.2 and 1.4) follow from our results. The supremacy of our conditions is clear since it requires less number of edges than similar existing degree related conditions to ensure the existence of Hamiltonian paths in a graph. Also it is proved in Section 3 that our conditions are better than that provided by Ore. Our results pose some new ideas especially for the degree related conditions for the hamiltonicity of graphs, the introduction of the parameter $\delta(u, v)$ being one of them. Since in Theorem 1.6 we have presented a sufficient condition for a graph to possess a Hamiltonian path, the natural extension of this research should be to find a similar condition for a graph to be Hamiltonian. Also, an interesting idea would be to weaken our requirements of satisfying the condition for all pairs of nonadjacent vertices and thereby, in effect, lower the number of edge-requirements by our condition.

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