Positive Solutions of a Class of Linear Systems

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ABSTRACT

A set of easily verifiable sufficient conditions are derived for the existence of a positive (componentwise) solution of a linear nonhomogeneous system of equations in which the coefficient matrix can be partitioned into submatrices with positive elements.

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1. INTRODUCTION

In a recent article the author [3] has derived a set of easily verifiable sufficient conditions for the existence of a unique solution vector with positive components of a linear system of the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1,2,\ldots,n,$$

where $a_{ii} > 0$, $b_i > 0$, $i = 1,2,\ldots,n$, and $a_{ij} \geqslant 0$, $i, j = 1,2,\ldots,n$, $i \neq j$. The purpose of the following is to achieve a generalization of the result of [3] and obtain sufficient conditions for the existence of a solution vector with positive components of a linear system of the form

$$\begin{bmatrix} A & B \\ C & -D \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \tag{1.1}$$

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where

$$\mathbf{x} = \text{col}\{x_1, \ldots, x_m\},$$

$$\mathbf{y} = \text{col}\{y_1, \ldots, y_{n-m}\},$$

$$\mathbf{f} = \text{col}\{f_1, \ldots, f_m\},$$

$$\mathbf{g} = \text{col}\{g_1, \ldots, g_{n-m}\},$$

$$A = (a_{ij}), \qquad i, j = 1, 2, \ldots, m,$$

$$B = (b_{ij}), \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n-m,$$

$$C = (c_{ij}), \qquad i = 1, 2, \ldots, n-m, \quad j = 1, 2, \ldots, m,$$

$$D = (d_{ij}), \qquad i, j = 1, 2, \ldots, n-m.$$

The problem of the existence of a solution vector $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ with positive components is of interest in a problem of persistence of multispecies population systems in ecology. We shall elaborate this aspect at the end of the article.

We note that one can claim that necessary and sufficient conditions for the existence of a componentwise positive solution of (1.1) can be obtained by means of Cramer's rule; however, such conditions are often not ecologically interpretable or even capable of meaningful interpretation at all, especially in applications. It is thus of some interest to derive a set of easily verifiable conditions for the existence of a componentwise positive solution of (1.1).

2. THE MAIN RESULT

The following notation will be used below: For square matrices of the same type $A = (a_{ij}) \geqslant B = (b_{ij})$ means that $a_{ij} \geqslant b_{ij}$ for all i, j. For column vectors $\mathbf{a} = (a_i) \geqslant \mathbf{b} = (b_i)$ means that $a_i \geqslant b_i$ for all i; $\mathbf{a} \gg \mathbf{b}$ means that $a_i > b_i$ for all i.

Let A_D be the diagonal matrix consisting of diagonal elements of any matrix A. The following preparation will be useful in formulating our result below: define

$$\mathcal{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \qquad Y = \begin{bmatrix} A_D & 0 \\ C & -D_D \end{bmatrix}. \tag{2.1}$$

We will assume that all the diagonal elements of A and D are positive; i.e.,

$$A_D^{-1} \text{ exists} \quad \text{and} \quad A_D^{-1} \geqslant 0, \tag{2.2}$$

$$D_D^{-1} \text{ exists} \quad \text{and} \quad D_D^{-1} \geqslant 0. \tag{2.3}$$

As a consequence of (2.2) and (2.3) it will follow that Y in (2.1) is invertible and

$$Y^{-1} = \begin{bmatrix} A_D^{-1} & 0 \\ D_D^{-1}CA_D^{-1} & -D_D^{-1} \end{bmatrix}, \tag{2.4}$$

with which we define the matrix X as follows:

$$X = Y^{-1}\begin{bmatrix} A - A_D & B \\ 0 & -D + D_D \end{bmatrix}. \tag{2.5}$$

It can be found from (2.5) that

$$X = \begin{bmatrix} A_D^{-1}(A - A_D) & A_D^{-1}B \\ D_D^{-1}CA_D^{-1}(A - A_D) & D_D^{-1}CA_D^{-1}B + D_D^{-1}(D - D_D) \end{bmatrix} \geqslant 0 \tag{2.6}$$

and

$$H = \begin{bmatrix} A & B \\ C & -D \end{bmatrix} = Y(I + X). \tag{2.7}$$

We can now formulate our result in the following:

THEOREM 2.1. *Suppose the following assumptions hold:*

(1) $\mathbf{c} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \gg 0.$
(2) *The diagonal elements of A and D are positive.*
(3) $A \geqslant 0,\ B \geqslant 0,\ C \geqslant 0,\ D \geqslant 0.$
(4) *For the vector* $\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{y}^{(0)} \end{bmatrix}$ *defined by* $Y^{-1}\mathbf{c}$ *we have*

$$\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{y}^{(0)} \end{bmatrix} = Y^{-1}\mathbf{c} \gg 0. \tag{2.8}$$

(5) *For the vector* $\begin{bmatrix} \mathbf{u}^{(0)} \\ \mathbf{v}^{(0)} \end{bmatrix}$ *defined by* $(I-X)Y^{-1}\mathbf{c}$ *we have*

$$\begin{bmatrix} \mathbf{u}^{(0)} \\ \mathbf{v}^{(0)} \end{bmatrix} = (I-X)Y^{-1}\mathbf{c} \gg 0. \tag{2.9}$$

Then the matrix H is invertible and is such that the solution of (1.1) *satisfies the positivity conditions*

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = H^{-1}\mathbf{c} \gg 0. \tag{2.10}$$

Furthermore, the unique positive solution $H^{-1}\mathbf{c}$ of (1.1) can be obtained as the limit of the convergent sequence

$$\mathbf{d}^{(n)} = \sum_{j=0}^{n} (-1)^j X^j Y^{-1}\mathbf{c}. \tag{2.11}$$

Proof. We first show that the spectral radius of X, $\rho(X) < 1$. Assumptions (4) and (5) imply that no row of $I-X$ can consist only of nonpositive elements. But by (2.6), off-diagonal entries of $I-X$ are nonpositive, so diagonal elements should be strictly positive. Now from (2.8), (2.9), and the non-positivity of off-diagonal entries of $I-X$, the generalized Hawkins-Simon theorem [5, p.7] asserts that $I-X$ is a P-matrix (i.e., all principal minors of $I-X$ are positive). So, by [5, Theorem 30, p. 25] it follows that $\rho(X) < 1$. So both $I-X$ and $I+X$ are invertible, and

$$\begin{aligned} (I+X)^{-1}Y^{-1}\mathbf{c} &= (I-X^2)^{-1}(I-X)Y^{-1}\mathbf{c} \\ &= \sum_{n=0}^{\infty} X^{2n}(I-X)Y^{-1}\mathbf{c} \gg 0. \end{aligned} \tag{2.12}$$

The invertibility of H follows from that of Y and $I+X$ and (2.12) shows that

$$H^{-1}\mathbf{c} = (Y(I+X))^{-1}\mathbf{c} = (I+X)^{-1}Y^{-1}\mathbf{c} \gg 0.$$

To prove convergence of the sequence $\{\mathbf{d}^{(n)}\}$ we define error vector $\mathbf{e}^{(n)}$ by

$$\mathbf{e}^{(n)} = H^{-1}\mathbf{c} - \mathbf{d}^{(n)} \tag{2.13}$$

Then

$$\begin{aligned}\|\mathbf{e}^{(n+1)}\| &= \left\| \sum_{j=n+2}^{\infty} (-1)^j X^j Y^{-1}\mathbf{c} \right\| \\ &= \left\| -X \sum_{j=n+1}^{\infty} (-1)^j X^j Y^{-1}\mathbf{c} \right\| \\ &= \|X\mathbf{e}^{(n)}\| = \|X^{n+1}\mathbf{e}^{(0)}\| \leqslant \|X^{n+1}\|\,\|\mathbf{e}^{(0)}\| \leqslant \|X\|^{n+1}\|\mathbf{e}^{(0)}\|.\end{aligned} \tag{2.14}$$

Since $\rho(X) < 1$, by a result in [4, p. 44] there is a norm such that $\|X\| < 1$. Hence for some γ, $0 < \gamma < 1$, we have $\|X\| = \gamma < 1$, which will imply that

$$\|\mathbf{e}^{(n+1)}\| \leqslant \gamma^{n+1}\|\mathbf{e}^{(0)}\| \tag{2.15}$$

Thus γ is an estimate for the convergence of the iterative method described above. ■

Remark 1. The vector sequences

$$\begin{bmatrix} \mathbf{x}^{(n)} \\ \mathbf{y}^{(n)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{u}^{(n)} \\ \mathbf{v}^{(n)} \end{bmatrix}$$

defined below are monotonically convergent to the solution vector $H^{-1}\mathbf{c}$:

$$\sum_{j=0}^{2n} (-1)^j X^j Y^{-1}\mathbf{c} = \begin{bmatrix} \mathbf{x}^{(n)} \\ \mathbf{y}^{(n)} \end{bmatrix} \downarrow H^{-1}\mathbf{c}, \tag{2.16}$$

$$\sum_{j=0}^{2n+1} (-1)^j X^j Y^{-1}\mathbf{c} = \begin{bmatrix} \mathbf{u}^{(n)} \\ \mathbf{v}^{(n)} \end{bmatrix} \uparrow H^{-1}\mathbf{c}. \tag{2.17}$$

Remark 2. If we replace conditions (1), (4), and (5) of the theorem by weaker inequalities, we shall again be able to apply the generalized Hawkins-Simon theorem and our theorem will hold true with weaker inequalities in place of (2.10). This remark is due to a suggestion of one of the referees.

3. AN APPLICATION IN MATHEMATICAL ECOLOGY

One of the problems currently studied by several authors in mathematical ecology is the problem of *persistence* of multispecies population systems (see for instance Gopalsamy [1]). A multispecies population system described by a nonnegative $\mathbf{w}:[0,\infty)\to R_n^+$ is said to be persistent if and only if there exist positive constants α_i, β_i $(i=1,2,\ldots,n)$ such that

$$w_i(0)>0 \;\Rightarrow\; 0<\alpha_i\leqslant w_i(t)\leqslant\beta_i, \qquad i=1,2,\ldots,n.$$

It has been shown by Gopalsamy [1] that a necessary condition for the persistence of a multispecies population system governed by the Lotka-Volterra equations

$$\frac{dx_i}{dt}=x_i\left\{f_i-\sum_{j=1}^{m}a_{ij}x_j-\sum_{j=1}^{n-m}b_{ij}y_j\right\}, \qquad i=1,2,\ldots,m,$$

$$\frac{dy_i}{dt}=y_i\left\{-g_i+\sum_{j=1}^{m}c_{ij}x_j-\sum_{j=1}^{n-m}d_{ij}y_j\right\}, \qquad i=1,2,\ldots,n-m, \tag{3.1}$$

is that there exists a positive solution of the linear system (1.1) as defined in the introduction. Since the existence of a positive solution of (1.1) is closely connected with the persistence of the system (3.1), our analysis of (1.1) is of some ecological significance.

All the conditions assumed in our Theorem 2.1 have ecological and intuitive interpretations. The positivity of the vectors $\mathbf{f},\mathbf{g}$ and the nonnegativity of the elements of the matrices A,B,C,D are directly related to the system (3.1). The strict positivity of the diagonal elements of A and D corresponds to the intraspecific competition of each species. Assumptions (4) and (5) correspond to the following: there is a maximum positive saturation level of each prey species in the absence of all other species, and corresponding to this there is a maximum positive saturation level of each predator species in the absence of all other predator species. Furthermore, there exists a positive minimum level of each prey species in the absence of competition among prey, and correspondingly there exists a positive minimum level of each predator species in the absence of competition among different predators. Our conditions (4) and (5) are formulated so as to guarantee the solutions of (3.1) to remain positively bounded below for all $t > 0$ provided $x_i(0) > 0$, $y_i(0) > 0$. It can also be shown that under the conditions of (3.1) the vectors corresponding to

$$x_i(0) > 0, \quad y_j > 0 \qquad (i = 1, \dots, m \quad j = 1, 2, \dots, n - m),$$

will have the property

$$\begin{bmatrix} \mathbf{x}_i^{(t)} \\ \mathbf{y}_i^{(t)} \end{bmatrix} \to \begin{bmatrix} \mathbf{x}^* \\ \mathbf{y}^* \end{bmatrix} \quad \text{as} \quad t \to \infty, \tag{3.2}$$

and this can be accomplished by a method similar to that in Gopalsamy and Ahlip [2]. We conclude with the remark that the conditions of our theorem are not only expressed directly in terms of the parameters of the system (1.1) and hence verifiable in a finite number of arithmetical steps (recall our remark about Cramer's rule in the introduction) but also have intuitive ecological significance indicating possible global behavior as in (3.2), which implies that the system (3.1) cannot oscillate indefinitely [absence of periodic solution of (3.1)].

4. AN EXAMPLE

We illustrate the above result and the iterative procedure by the following example:

$$\begin{aligned} 10x_1 + x_2 + y_1 &= 14, \\ x_1 + 10x_2 + y_1 &= 14, \\ 10x_1 + x_2 - 2y_1 &= 5. \end{aligned}$$

In our notation,

$$A = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = [10, 1]$$

$$D = [2], \qquad \mathbf{f} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}, \qquad \mathbf{g} = [5].$$

It can be easily verified that the above system of equations satisfies the conditions of the theorem. So the system must have a positive solution. For this problem it can be computed that

$$X = \begin{bmatrix} 0 & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{20} & \frac{1}{2} & \frac{11}{20} \end{bmatrix}$$

and the spectral radius of X is $\frac{13}{20}$. So the iteration scheme converges at least at a rate equal to $\frac{13}{20} + \varepsilon$ for any $\varepsilon > 0$ (note that it is not necessary for all row sums and column sums of X to be less than 1).

The successive approximations $\mathbf{d}^{(n)}$ of the solution vector $H^{-1}\mathbf{c}$, computed by the iterative procedure, are given in Table 1. It is found that the exact solution vector is

$$\mathbf{d}^* = \begin{bmatrix} 1.0 \\ 1.0 \\ 3.0 \end{bmatrix}.$$

The error vectors $\mathbf{e}^{(k)}$ corresponding to the approximate solution vectors $\mathbf{d}^{(k)}$, shown in the Table 2, justify the validity of (2.15).

Note that (2.15) is satisfied with sign of equality in this example, and this is due to the fact that initial error vector $\mathbf{e}^{(0)}$ is an eigenvector of X corresponding to the spectral radius.

TABLE 1

SUCCESSIVE APPROXIMATIONS TO THE SOLUTION

$\mathbf{d}^{(k)}$	$K=0$	1	2	3	4	5
$\mathbf{x}_1^{(k)}$	1.40	0.74	1.169	0.890	1.071	0.953
$\mathbf{x}_2^{(k)}$	1.40	0.74	1.169	0.890	1.071	0.953
$\mathbf{y}_1^{(k)}$	5.20	1.57	3.929	2.396	3.393	2.745

TABLE 2

SUCCESSIVE ERROR VECTORS

$\mathbf{e}^{(k)}$	$K=0$	1	2	3	4	5
$\mathbf{e}_1^{(k)}$	− 0.40	0.260	− 0.169	0.110	− 0.071	0.047
$\mathbf{e}_2^{(k)}$	− 0.40	0.260	− 0.169	0.110	− 0.071	0.047
$\mathbf{e}_3^{(k)}$	− 2.20	1.430	− 0.929	0.604	− 393	0.255

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