Zu Eigenwerten, -Vektoren und Hauptachsentransformation

Aufgabe 2)

a) A: $\lambda_1 = -2 \to (1,-4)$ $\quad \lambda_2 = 3 \to (1,1)$

B: $\lambda_1 = 4 \to (-1,1,0)$, $\lambda_2 = -8 \to (0,0,1)$, $\lambda_3 = 6 \to (1,1,0)$

C: $\lambda_1 = 2 \to (1,0,0)$, $\lambda_{2,3} = 1 \to (1,-1,1)$

D: $\lambda_{1,2} = 1 \to (1,1,0), (-2,0,1)$; $\lambda_3 = 7 \to (1,-1,2)$

$\mathfrak{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad ax^2 + bxy + cy^2 + dx + ey + f = 0$; $F(x,y) = 0$

$\mathfrak{b} = \begin{pmatrix} d \\ e \end{pmatrix}$ $\qquad \mathfrak{r}^T A \mathfrak{r} + \mathfrak{b}^T \mathfrak{r} + f = 0$ mit $A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$

Transform.: $\mathfrak{u} = \begin{pmatrix} u \\ v \end{pmatrix} \to \mathfrak{u}^T \Lambda \mathfrak{u} + (\underbrace{S^T \mathfrak{b}}_{\tilde{\mathfrak{b}}})^T \mathfrak{u} + f = 0$ mit $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

λ_1, λ_2 .. Eigenwerte von A

S ... Transform. Matrix (Drehmatrix) : $\lambda_1 u^2 + \lambda_2 v^2 + \tilde{\mathfrak{b}}^T \mathfrak{u} + f = 0$

Bsp: $5x^2 + 8xy + 5y^2 - 20\sqrt{2}\,x - 16\sqrt{2}\,y + 31 = 0$

$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, $\mathfrak{b} = \begin{pmatrix} -20\sqrt{2} \\ -16\sqrt{2} \end{pmatrix}$, $f = 31$

$\lambda_1 = 9 \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 1 \to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\to$ Drehmatrix $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$9u^2 + v^2 + \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -20\sqrt{2} \\ -16\sqrt{2} \end{pmatrix} \right]^T \begin{pmatrix} u \\ v \end{pmatrix} + 31 = 0$$

Ellipse

$$9u^2 + v^2 - 36u + 4v + 31 = 0$$

$$9(u-2)^2 + (v+2)^2 - 9 = 0 \to (u-2)^2 + \frac{(v+2)^2}{9} = 1$$

oder: $\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} u - \frac{1}{\sqrt{2}} v \\ \frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}} v \end{pmatrix}$ oben einsetzen und zusammenfassen (ist mühsamer!)

4a) $\lambda_1 = -\frac{1}{2} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = \frac{5}{2} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix} \to S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\qquad -\frac{1}{2} u^2 + \frac{5}{2} v^2 - 5 = 0$

$$-\frac{u^2}{10} + \frac{v^2}{2} = 1$$ Hyperbel

b) $\lambda_1 = 5$, $\lambda_2 = 1 \to S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ $\qquad 5u^2 + v^2 = 1$ Ellipse

c) $\lambda_1 = 0$, $\lambda_2 = 5 \to S = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ $\qquad 5v^2 - \frac{6}{\sqrt{5}} u + \frac{12}{\sqrt{5}} v + 5 = 0$

Parabel

4 d) $2x^2 + 4xy + y^2 + \sqrt{2}\,x - 1 = 0$

$A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $\lambda_1 = \frac{3+\sqrt{17}}{2}$ $EV_1 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{17}}{4} \end{pmatrix}$

$\lambda_2 = \frac{3-\sqrt{17}}{2}$ $EV_2 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{17}}{4} \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{17}}{4} & \frac{1+\sqrt{17}}{4} \end{pmatrix} \to \begin{pmatrix} \frac{1+\sqrt{17}}{4} & 1 \\ -1 & \frac{1+\sqrt{17}}{4} \end{pmatrix} \to S = \frac{\sqrt{8}}{\sqrt{\sqrt{17}+17}} \begin{pmatrix} \frac{1+\sqrt{17}}{4} & 1 \\ -1 & \frac{1+\sqrt{17}}{4} \end{pmatrix}$$

$\lambda_1 > 0$, $\lambda_2 < 0$ Hyperbel

e) $\lambda_1 = \frac{-5+\sqrt{5}}{2} \to \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$ $\lambda_2 = \frac{-5-\sqrt{5}}{2} \to \begin{pmatrix} 1 \\ \frac{-\sqrt{5}+1}{2} \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \to \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \to S = \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

f) $x^2 - 4xy + y^2 - 2x + 4y - 2 = 0$ $\lambda_1 = 3$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda_2 = -1$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $S = \frac{1}{2}\sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Hyperbel $3u^2 - v^2 - 3\sqrt{2}\,u + \sqrt{2}\,v - 2 = 0$

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6) $y_1 = 3x_1 + x_2$ $\varphi(\mathbb{R}^2) \to \mathbb{R}^3$ läßt sich interpretieren als

$y_2 = x_1 + 4x_2$ $y = x_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ Ebene in $\mathbb{R}^3$

$y_3 = x_1 + 5x_2$

6.1. g: $x_1 + x_2 = 0 \to x_2 = -x_1$ $y_1 = 3x_1 - x_1 = 2x_1$, $y_2 = -3x_1$, $y_3 = -4x_1$

$y = x_1 \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \cong t \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$ Gerade in $\mathbb{R}^3$

6.2. ε: $y_1 - y_2 + y_3 = 2 \to (3x_1 + x_2) - (x_1 + 4x_2) + (x_1 + 5x_2) = 2$

$3x_1 + 2x_2 = 2$ Gerade in $\mathbb{R}^2$